DISTANCE LEARNING CENTRE

Ahmadu Bello University
Zaria, Nigeria

MATH201:
Mathematical Methods I

Course Material

Programme Title: B.Sc. Computer Science

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Course Study Guide

Course Information

Course Code: MATH201

Course Title: Mathematical Methods I

Credit Units: Three (3)

Year of Study: 2

Semester: First

Course Introduction and Description

Introduction:

You are welcome to MATH 201, this is a 3-credit unit year 2, first semester course. I will be your guide in the span of this course, you should feel free to ask questions when in a form of difficulties relating this course, and my door is always open.

This course in the continuation of introduction to differential and integral calculus taught in Math 105. In this course, we shall discuss the basic applications of calculus; these applications include, rate of change, area under curve, radius of curvature, minimum, maximum and saddle point. At the end of this course, you should be able to solve basic differentiation, integration, partial differentiation, equation of tangent and normal, minimum and maximum points for one or more variables.

Description:

This course title MATH201 called Mathematical methods. I shall discuss with you, the different applications of integral and differential calculus to day to day activities. The course is divided into three modules of 8 study sessions.

i. COURSE PREREQUISITES

You should note that although this course has no subject pre-requisite, you are expected to have:

1. Satisfactory level of English proficiency

2. Basic Computer Operations proficiency
3. Online interaction proficiency
4. Web 2.0 and Social media interactive skills
5. MATH105

ii. COURSE LEARNING RESOURCES

i. Course Textbooks

Robert, C., Wrede and Murray, R., Spiegel, (2002), Advanced Calculus, 2nd Edition, Schaum's Outline Series McGRAW-HILL, New York Chicago San Francisco Lisbon London Madrid Mexico City Milan New Delhi San Juan Seoul Singapore Sydney Toronto.

"Course code: Course Title, National Open University of Nigeria, 2014 at http://www.nou.edu.ng/index.htm#

Thomas G.B and Finney R. L (1982) Calculus and Analytic Edition, Addison-Wesley Publishing Company, Would student series Edition, London, Sydrey, Tokyo, Manila, Reading.

John Bird (2006), Higher Engineering Mathematics, 5th Edition, Linacre House, Jordan Hill, Oxford OX2 8DP 30 Corporate Drive, Suite 400, Burlington, MA01803, USA.

iii. COURSE OBEJCTIVES AND OUTCOMES

After you are done studying this course, you should be able to:

1.Solve basic differentiations and integrations in mathematical modelling

2. Explain the applications of partial differentiation

3. Workout the Taylor's series expansion and radius of curvature of a curve.

4. Compute the minima, maxima and saddle point of functions of two or more variables.

iv. ACTIVITIES TO MEET COURSE OBJECTIVES

Specifically, this course shall comprise of the following activities:

1. Studying courseware

2. Listening to course audios
3. Watching relevant course videos
4. Field activities, industrial attachment or internship, laboratory or studio work (whichever is applicable)
5. Course assignments (individual and group)
6. Forum discussion participation
7. Tutorials (optional)
8. Semester examinations (CBT and essay based).

v. TIME (TO COMPLETE SYLABUS/COURSE)

To cope with this course, you would be expected to commit a minimum of 3 hours weekly for the Course.

viii. GRADING CRITERIA AND SCALE

Grading Criteria

A. Formative assessment

Grades will be based on the following:

Individual assignments/test (CA 1,2 etc)	20
Group assignments (GCA 1, 2 etc)	10
Discussions/Quizzes/Out of class engagements etc	10

B. Summative assessment (Semester examination)

CBT based	30
Essay based	30
TOTAL	**100%**

C. Grading Scale:

A = 70-100

B = 60 – 69

C = 50 - 59

D = 45-49

F = 0-44

D. Feedback

Courseware based:

1. In-text questions and answers (answers preceding references)
2. Self-assessment questions and answers (answers preceding references)

Tutor based:

1. Discussion Forum tutor input
2. Graded Continuous assessments

Student based:

1. Online programme assessment (administration, learning resource, deployment, and assessment).

IX. LINKS TO OPEN EDUCATION RESOURCES

OSS Watch provides tips for selecting open source, or for procuring free or open software.

SchoolForge and SourceForge are good places to find, create, and publish open software. SourceForge, for one, has millions of downloads each day.

Open Source Education Foundation and Open Source Initiative, and other organisation like these, help disseminate knowledge.

Creative Commons has a number of open projects from Khan Academy to Curriki where teachers and parents can find educational materials for children or learn about Creative Commons licenses. Also, they recently launched the School of Open that offers courses on the meaning, application, and impact of "openness."

Numerous open or open educational resource databases and search engines exist. Some examples include:

- OEDb: over 10,000 free courses from universities as well as reviews of colleges and rankings of college degree programmes

- Open Tapestry: over 100,000 open licensed online learning resources for an academic and general audience
- OER Commons: over 40,000 open educational resources from elementary school through to higher education; many of the elementary, middle, and high school resources are aligned to the Common Core State Standards
- Open Content: a blog, definition, and game of open source as well as a friendly search engine for open educational resources from MIT, Stanford, and other universities with subject and description listings
- Academic Earth: over 1,500 video lectures from MIT, Stanford, Berkeley, Harvard, Princeton, and Yale
- JISC: Joint Information Systems Committee works on behalf of UK higher education and is involved in many open resources and open projects including digitising British newspapers from 1620-1900!

Other sources for open education resources

Universities

- The University of Cambridge's guide on Open Educational Resources for Teacher Education (ORBIT)
- OpenLearn from Open University in the UK

Global

- Unesco's searchable open database is a portal to worldwide courses and research initiatives
- African Virtual University (http://oer.avu.org/) has numerous modules on subjects in English, French, and Portuguese
- https://code.google.com/p/course-builder/ is Google's open source software that is designed to let anyone create online education courses
- Global Voices (http://globalvoicesonline.org/) is an international community of bloggers who report on blogs and citizen media from around the world, including on open source and open educational resources

Individuals (which include OERs)

- Librarian Chick: everything from books to quizzes and videos here, includes directories on open source and open educational resources
- K-12 Tech Tools: OERs, from art to special education
- Web 2.0: Cool Tools for Schools: audio and video tools
- Web 2.0 Guru: animation and various collections of free open source software
- Livebinders: search, create, or organise digital information binders by age, grade, or subject (why re-invent the wheel?)

X. ABU DLC ACADEMIC CALENDAR/PLANNER

	PERIOD											
Semester	**Semester 1**				**Semester 2**					**Semester 3**		
Activity	**JAN**	**FEB**	**MAR**	**APR**	**MAY**	**JUN**	**JUL**	**AUG**	**SEPT**	**OCT**	**NOV**	**DEC**
Registration												
Resumption												
Late Registn.												
Facilitation												
Revision/ Consolidation												
Semester Examination												

N.B: - All Sessions commence in January

- 1 Week break between Semesters and 6 Weeks vocation at end of session.

- Semester 3 is OPTIONAL (Fast-tracking, making up carry-overs & deferments)

ix. COURSE STRUCTURE AND OUTLINE

Course Structure

WEEK/DAYS	MODULE	STUDY SESSION	ACTIVITY
Week1	**MODULE 1 APPLICATIONS OF CALCULUS**	**Study Session 1:** Title: Revision of techniques of differentiation	1. Read Courseware for the corresponding Study Session. 2. View the Video(s) on this Study Session 3. Listen to the Audio on this Study Session 4. View any other Video/U-tube (address/sitehttp://bit.ly/2JCa6oQ , http://bit.ly/2YYm6Gs , http://bit.ly/2YYm6Gs , http://bit.ly/2NXLfQE , http://bit.ly/2GdfZGS) 5. View referred OER (address/site) 6. View referred Animation (Address/Site) 7. Read Chapter/page of Standard/relevant text. 8. Any additional study material 9. Any out of Class Activity
Week 2		**Study Session 2** Title: Successive derivatives, Leibniz theorem, Taylor's and Maclaurin's	1. Read Courseware for the corresponding Study Session. 2. View the Video(s) on this Study Session 3. Listen to the Audio on this Study Session 4. View any other Video/U-tube (address/sitehttp://bit.ly/2NXVKmM , http://bit.ly/2JySVV4 , http://bit.ly/2JOfnbR ,

		series	http://bit.ly/30AnEGY , http://bit.ly/2XLMPth , http://bit.ly/2Ya4QRi) 5. View referred OER (address/site) 6. View referred Animation (Address/Site) 7. Read Chapter/page of Standard/relevant text. 8. Any additional study material 9. Any out of Class Activity
Week 3& 4		**Study Session 3** Title: Tangents and Normal and radius of curvature	1. Read Courseware for the corresponding Study Session. 2. View the Video(s) on this Study Session 3. Listen to the Audio on this Study Session 4. View any other Video/U-tube (address/sitehttp://bit.ly/2xStlUA , http://bit.ly/30DgCRI , http://bit.ly/2SiOOzj , http://bit.ly/2XSvMpH , http://bit.ly/2Gg38no , http://bit.ly/2O0Z80j , http://bit.ly/2Ggklgp , http://bit.ly/2JN9U4O , http://bit.ly/2JDv0nH) 5. View referred OER (address/site) 6. View referred Animation (Address/Site) 7. Read Chapter/page of Standard/relevant text. 8. Any additional study material 9. Any out of Class Activity

Week 5	**MODULE 2: Integral Calculus**	**Study Session 1** Title: Methods of integration	1. Read Courseware for the corresponding Study Session. 2. View the Video(s) on this Study Session 3. Listen to the Audio on this Study Session 4. View any other Video/U-tube (address/sitehttp://bit.ly/2XR9orz , http://bit.ly/32vroeN , http://bit.ly/2xSfDkm , http://bit.ly/2YYVDZw , http://bit.ly/2Y8cbB1 ,http://bit.ly/2XRa14F , http://bit.ly/32skW8d , http://bit.ly/2Y4FjsJ , http://bit.ly/2LQUDTs , http://bit.ly/2JDnnNX , http://bit.ly/2XWhejI , http://bit.ly/32wz7co . http://bit.ly/2XRnLRS , http://bit.ly/2GhM40f , http://bit.ly/32xRihO , http://bit.ly/2LVEWuf , http://bit.ly/2JNb6VQ , http://bit.ly/2LVBxeL) 5. View referred OER (address/site) 6. View referred Animation (Address/Site) 7. Read Chapter/page of Standard/relevant text. 8. Any additional study material 9. Any out of Class Activity
Week 6		**Study Session 2** Title: Definite	1. Read Courseware for the corresponding Study Session. 2. View the Video(s) on this Study Session 3. Listen to the Audio on this Study Session

		integrals, Area enclosed by a plane curve	4. View any other Video/U-tube (address/sitehttp://bit.ly/2JPFaQX , http://bit.ly/2GeipFm , http://bit.ly/2Y6eFQu , http://bit.ly/2XJItCQ , http://bit.ly/2xT2Uy1 , http://bit.ly/2XTJ7cr , http://bit.ly/32wfy44) 5. View referred OER (address/site) 6. View referred Animation (Address/Site) 7. Read Chapter/page of Standard/relevant text. 8. Any additional study material 9. Any out of Class Activity
Week 7& 8		**Study Session 3** Title: Improper integrals	1. Read Courseware for the corresponding Study Session. 2. View the Video(s) on this Study Session 3. Listen to the Audio on this Study Session 4. View any other Video/U-tube (address/sitehttp://bit.ly/2xT4Q9L , http://bit.ly/2JLQuNR , http://bit.ly/2Gcn4HB , http://bit.ly/30EJjOy) 5. View referred OER (address/site) 6. View referred Animation (Address/Site) 7. Read Chapter/page of Standard/relevant text. 8. Any additional study material 9. Any out of Class Activity

Week 9 & 10	**MODULE 3: Partial Differentiation**	**Study Session 1** Title: Partial derivatives and Jacobians	1. Read Courseware for the corresponding Study Session. 2. View the Video(s) on this Study Session 3. Listen to the Audio on this Study Session 4. View any other Video/U-tube (address/sitehttp://bit.ly/2LpBg4y , http://bit.ly/2Y9yshM , http://bit.ly/2Y9I0cA , http://bit.ly/2LpgIZE , http://bit.ly/32upBXj , http://bit.ly/30BFidq , http://bit.ly/32x9sQL) 5. View referred OER (address/site) 6. View referred Animation (Address/Site) 7. Read Chapter/page of Standard/relevant text. 8. Any additional study material 9. Any out of Class Activity
Week 11 & 12		**Study Session 2** Title: Lagrange's multiplier, differentials and linear approximation	1. Read Courseware for the corresponding Study Session. 2. View the Video(s) on this Study Session 3. Listen to the Audio on this Study Session 4. View any other Video/U-tube (address/site http://bit.ly/2GaLyRN , http://bit.ly/2LpC5ua , http://bit.ly/2NZYYGG , http://bit.ly/2XXsD33 , http://bit.ly/30Bfj60) 5. View referred OER (address/site)

			6. View referred Animation (Address/Site) 7. Read Chapter/page of Standard/relevant text. 8. Any additional study material 9. Any out of Class Activity
Week 13		**REVISION/TUTORIALS (On Campus or Online) & CONSOLIDATION WEEK**	
Week 14 & 15		**SEMESTER EXAMINATION**	

Course Outline

MODULE 1: Applications of Calculus

Study Session 1: Revision of techniques of differentiation

StudySession2:Successive derivatives, Leibniz theorem, Taylor's and Maclaurin's series

Study Session 3: Tangents and Normal and radius of curvature

MODULE 2: Integral Calculus

Study Session 1: Methods of integration

Study Session 2: Definite integrals, Area enclosed by a plane curve

Study Session 3: Improper integrals

MODULE 3: Partial Differentiation

Study Session 1: Partial derivatives and Jacobians

Study Session 2: Lagrange's multiplier, differentials and linear approximation

Study Modules

MODULE 1: Applications of Calculus

Content

Study Session 1: Revision of techniques of differentiation
Study Session 2: Successive derivatives, Leibniz theorem, Taylor's and Maclaurin's series.
Study Session 3: Tangents and Normal and radius of curvature

Study Session 1

Revision of Techniques of Differentiation

Section and Subsection Headings:

Introduction
1.0 Learning Outcomes
2.0 Main Content
  2.1- Sum, Difference, power, Product and Chain Rules
  2.2- Implicit Differentiation
  2.3- Parametric differentiation
3.0Tutor Marked Assignments (Individual or Group assignments)
4.0Study Session Summary and Conclusion
5.0Additional Activities (Videos, Animations & Out of Class activities)
6.0In-text Question Answers
7.0 References/Further Readings

Introduction:

You are welcome to our first study module of this course, study session one will focus on the basic differential techniques taught in Math105 in your first year.

These techniques serve important role in understanding rate of change, acceleration of a body and traffic flow. We shall also discuss the sum and difference rule, product rule, chain rule, implicit differentiation and parametric differentiation. Once again you are welcome.

1.0 Study Session Learning Outcomes

After you are done studying this session, you should to be able to:

1. Find the derivative of any chained functions
2. Obtains the first derivative of any given function
3. Recognize the various techniques of differentiation.

2.0 Main Content

2.1 Sum, Difference, power, Product and Chain Rules

If f and g are differentiable functions, then

(i) $\frac{d}{dx}(f(x)+g(x))=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)=f'(x)+g'(x)$ (sum rule)

(ii) $\frac{d}{dx}(f(x)-g(x))=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)=f'(x)-g'(x)$ (difference rule)

(iii) $\frac{d}{dx}\big(cf(x)\big)=c\frac{d}{dx}\big(f(x)\big)=cf'(x)$

(iv) $\frac{d}{dx}(f(x)g(x))=f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)=f(x)g'(x)+g(x)f'(x)$ (product rule)

(v) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$ (quotient rule)

(vi) If $y(x)=f(g(x))$, where $f(x)$ and $g(x)$ may or may not be the same type of function. Then the derivative

$$\frac{d}{dx}\big(y(x)\big)=\frac{d}{dg(x)}\big(y(x)\big)\times\frac{d}{dx}g(x)$$

(vii) If $y=x^n$ where $n\epsilon\mathbb{R}$, then $\frac{d}{dx}(x^n)=nx^{n-1}$

These three rules can be easily proved using the first principle approach discussed in Math105.

Table 1 below gives the derivative of basic functions which would be considered in this session.

Table 1.1.1: Differentiation of basic functions

	Function	Derivative
1.	$y = x^n : n \in \mathbb{R}$	$y' = nx^{n-1}$
2.	$y = sin\ ax : a \in \mathbb{R}$	$y' = acos\ ax$
3.	$y = cos\ ax : a \in \mathbb{R}$	$y' = -asin\ ax$
4.	$y = e^{ax} : a \in \mathbb{R}$	$y' = ae^{ax}$
5.	$y = alog_e x$ $= ln\ x$	$y' = \frac{a}{x}$
6.	$y = log_a x$	$y' = \frac{1}{x} log_a e$
7.	$y = a^x$	$y' = a^x log_e a$
8.	$y = tan^{-1} x$	$y' = \frac{1}{1 + x^2}$
9.	$y = sin^{-1} x$	$y' = \frac{1}{\sqrt{1 - x^2}}$
10.	$y = \sqrt{x}$	$y' = \frac{1}{2\sqrt{x}}$

2.1.1 Examples on Sum, difference and product rule

Example 1: If $f(x) = 3x^4 + 2log_e x - \sin x$, find $f'(x)$

Solution:

Using the sum rule and difference,

$$f'(x) = \frac{d}{dx} 3x^4 + \frac{d}{dx} 2log_e x - \frac{d}{dx} \sin x$$

We have;

$f'(x) = 12x^3 + \frac{2}{x} - \cos x$ (by the use of Table 1)

Example 2: If $f(x) = 10x^4 - 4sinx + e^{2x}$, find $f'(x)$

Solution:

Using the sum and difference rules,

$f'(x) = \frac{d}{dx}10x^4 - 4\frac{d}{dx}cosx + \frac{d}{dx}e^{2x}$ using the sum and difference rules

We have;

$f'(x) = 40x^3 + 4\sin x + 2e^x$ (by the use of Table 1.1.1)

Example 3: Find $\frac{dy}{dx}$ if $y = 2xe^{3x}$

Solution

Since the function y is a combination of exponential function and linear function, the derivative can easily be obtained by you using the product rule as follows:

Let $u = 2x, v = e^{3x}$,

implies that $\frac{du}{dx} = 2$, and $\frac{dv}{dx} = 3e^{3x}$ (from Table 1.1.1)

using product rule;

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$= 6xe^{3x} + 2e^{3x}$

$\frac{dy}{dx} = 2e^{3x}(1 + 3x)$ (by collecting the common terms)

Example 4: Given $y = xsin^{-1}(x)$, find $\frac{dy}{dx}$

Solution:

You should use the product rule,

Let $u = x, v = sin^{-1}(x)$,

So that $\frac{du}{dx} = 1$, and $\frac{dv}{dx} =$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{1 - x^2}} + sin^{-1}(x)$$

2.1.2 Examples on quotient rule

Example 5: Compute the derivative of $y(x) = \frac{x^2}{2e^{4x}}$

Solution

Let $u = x^2$, $v = 2e^{4x}$,

Then, $\frac{du}{dx} = 2x, \frac{dv}{dx} = 8e^{4x}$

By the use of quotient rule,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{4xe^{4x} - 8\boldsymbol{x^2}e^{4x}}{4e^{8x}}$$

Example 6: If $y(x) = \ln\frac{4\sin(2x^2)}{(x+1)^5}$

Solution

$$\ln\frac{4\sin(2x^2)}{(x+1)^5} = \ln 4\sin(2x^2) - \ln(x+1)^5$$

So that

$$\frac{dy}{dx} = \frac{d}{dx}\ln[4\sin(2x^2)] - \frac{d}{dx}\ln(x+1)^5$$

$$= \frac{\frac{d}{dx}[4\sin(2x^2)]}{[4\sin(2x^2)]} - 5\frac{\frac{d}{dx}(x+1)}{(x+1)}$$

$$\frac{dy}{dx} = \frac{[16x\cos(2x^2)]}{[4\sin(2x^2)]} - \frac{5}{(x+1)}$$

On simplifying

$$\frac{dy}{dx} = 4xcot(2x^2) - \frac{5}{(x+1)}$$

2.1.2 Examples on chain rule

Example 7: Determine the differential coefficient of $y = \sqrt{3x^2 + 10x - 1}$

Solution

Let $u = 3x^2 + 10x - 1$, so that $y = \sqrt{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

By the use of Table 2,

$\frac{dy}{du} = \frac{1}{2\sqrt{u}}, \ \frac{du}{dx} = 6x + 10$

$$\frac{dy}{dx} = \frac{6x + 10}{2\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{6x + 10}{2\sqrt{3x^2 + 10x - 1}}$$

Example 8: Find the derivative of the function $y = e^{\sin(3x^2-1)}$

Solution:

Let $u = \sin(3x^2 - 1)$, so that $y = e^u$,

Let us use that chain rule,

$\frac{du}{dx} = 6x\, cos(3x^2 - 1), \ \frac{dy}{du} = e^u$

$$\frac{dy}{dx} = 6x\, cos(3x^2 - 1)e^{\sin(3x^2-1)}$$

In-text Question 1: *What is the derivative of* $4x^{1/2}$*?*

Answer: $\frac{2}{\sqrt{x}}$

2.2. Implicit Differentiation

In the major topic above, we see that all functions are given in explicit forms, in this part, we shall study the derivative of function which are not explicitly presented. However, there are functions such as

$$x^2y = 2xy + 100$$

This type of function is expressed implicitly. To obtain an explicit expression of an implicit expression you make y subject of formula.

For instance,

$2x^2 + 3y = 6$ (making y subject of formula)

yields $y = \frac{2x^2 - 6}{3}$

where the derivative can now be obtained as;

$$\frac{dy}{dx} = \frac{4x6}{3}$$

However,we observe that, these implicit functions are not easy to solve for y. Example of such functions are

(1)$x^2 + 5xy^4 + y^3x + 4x^3 = 2$

(2)$3x^3 + 2xy^2 + 10x^2 + 4y = 1$

(2)$e^{2xy} + xy^2 + 5xy^4 + 4y = 1$

In the above, it is not possible to solve for y, but the functions can be differentiated by the method of known as *implicit differentiation*. Appropriate applications of the rules for differentiations would be able to carry out to obtain the implicit differentiation. The next questions that comes to your mind should be "what is implicit differentiation" this question is best answered byyou followingthese examples:

Example 9: find $\frac{dy}{dx}$ of $3xy + 2x^2 + 2y^3 = 23000$

Solution:

Differentiation both sides of the equation with respect to x

$$\frac{d}{dx}(3xy) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(2y^3) = \frac{d}{dx}(23000)$$

Taking the derivatives one by one and considering product rule in cases where x and y are multiplied together:

$$3\left(x\frac{dy}{dx}+y\right)+4x+6y^2\frac{dy}{dx}=0$$

$\frac{dy}{dx}(3x+6y^2)=-(3y+4x)$ (collecting the like terms)

$\frac{dy}{dx}=\frac{-(3y+4x)}{(3x+6y^2)}$ (which is the required solution)

Example 10: Find $\frac{dy}{dx}$ of $x^3y^2-\sin 3x^2y=\tan y^2$ at $(1,0)$

Solution:

Differentiation both sides of the equation with respect to x

$$\frac{d}{dx}(x^3y^2)-\frac{d}{dx}(\sin(3x^2y))=\frac{d}{dx}(\tan y^2)$$

Using the appropriate techniques of differentiation,

$2x^3y^2\frac{dy}{dx}+3x^2y^2-3\left(x^2\frac{dy}{dx}+2xy\right)\cos(3x^2y)=2y\frac{dy}{dx}sec^2(y^2)$

Collecting the like terms,

$$\frac{dy}{dx}\left(2x^3y^2-3x^2\cos(3x^2y)-2ysec^2(y^2)\right)=6xy\cos(3x^2y)-3x^2y^2$$

$$\frac{dy}{dx}=\frac{6xy\cos(3x^2y)-3x^2y^2}{\left(2x^3y^2-3x^2\cos(3x^2y)-2ysec^2(y^2)\right)}$$

Substituting $x=1, y=0$ in the solution obtained for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx}=\frac{6(1)(0)\cos(3(1)^2(0))-3(1)^2(0)^2}{\left(2(1)^3(0)^2-3(1)^2\cos\left(3(1)^2(0)\right)-2(0)sec^2(0)\right)}$$

$$\frac{dy}{dx}=\frac{0}{-3}=0$$

In-text Question 2: What is the *derivative of* $3xy^2=100$ *at* $(1,1)$ *?*

Answer: $-\frac{1}{2}$

2.3 Parametric Differentiation

In this part, we observe that both the dependent and independent variable are function of a new parameter (say t). That is, $y = y(t)$ and $x = x(t)$, to find the derivative of such function, we should follow the following procedures:

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$ (called parametric differentiation)

Example 11: If $x = cos\,(2at)$, $y = sin^2(at)$, find $\left.\frac{dy}{dx}\right|_{t=0}$

Solution

$$\frac{dx}{dt} = \frac{d}{dt}[cos\,(2at)] = -2asin\,(at)$$

$$\frac{dy}{dt} = \frac{d}{dt}[sin^2(at)] = 2\,\mathrm{asin}(at)\cos(at)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{2\,\mathrm{asin}(at)\cos(at)}{-2asin\,(at)}$$

$$\frac{dy}{dx} = -\cos(at)$$

at $t = 0$, we have;

$$\left.\frac{dy}{dx}\right|_{t=0} = -\cos(a \times 0)$$

$$\left.\frac{dy}{dx}\right|_{t=0} = -1$$

Example 12: If $x = e^{2t^2}$, $y = sin(3t)$, find $\frac{dy}{dx}$

Solution

Following similar approach,

$$\frac{dx}{dt} = \frac{d}{dt}\left[e^{2t^2}\right] = 4te^{2t^2}$$

$\frac{dy}{dt} = \frac{d}{dt}[sin(3t)] = 3\cos(3t)$ (by simple differentiation with respect to t)

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{3\cos(3t)}{4te^{2t^2}}$$

In-text Question 3: *Compute the derivative of* $y = 3t^2 - t,\ x = e^{-5t}$ *at* $t = 0$.

Answer: $\frac{1}{5}$

3.0Tutor Marked Assignments (Individual or Group)

1. If $x = sin\,(5t^2), y = cos^2(at)$, find $\left.\frac{dy}{dx}\right|_{t=0}$

2. Find $\frac{dy}{dx}$ of $x^5y^2 - 3x^2y = e^{y^2}$

3. If $f(x) = x^7 - 4sin(x^5) + e^{2x^2}$, find $f'(x)$

4.0Conclusion/Summary

In this study session, we didreview the basic techniques of differentiation studied in Math105. These techniques form the basis for different physical situations such as velocity of a body, acceleration of a moving body and skin-friction. The basic differential techniques presented are; the sum and difference rule, product rule, quotient rule, chain rule, implicit differentiation and parameter tic differentiation. It is expected that every student should be able to perform all the stated basic rules.

5.0 Self-Assessment Questions

1. Find the derivative of $\sin(x^2 - 4x)$.
2. Differentiate the function $y = x\log(\sin x)$.

Answer to Self-Assessment Questions:

1. $(2x - 4)\cos(x^2 - 4x)$
2. $\frac{dy}{dx} = \log(\sin x) + x\tan x$

6.0 Additional Activities (Videos, Animations & Out of Class activities)

a. Visit U-tube http://bit.ly/2JCa6oQ , http://bit.ly/2YYm6Gs , http://bit.ly/2YYm6Gs , http://bit.ly/2NXLfQE , http://bit.ly/2GdfZGS. Watch the video & summarise in 1 paragraph

b. View the animation on techniques of differentiation and critique it in the discussion forum

c. Take a walk and engage any 3 students on the differenttechniques of differentiation; In 2 paragraphs summarise their opinion of the discussed topic.

7.0 References/Further Readings

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Study Session 2

Successive Derivatives, Leibniz Theorem, Taylor's And Maclaurin's Series

Section and Subsection Headings:

Introduction:

In thissession, we shall spread our coast in discuss some applications of differential calculus which forms basic tools in solving real life mathematical problems. They include successive derivatives, Leibniz theorem, Taylor's and Maclaurin's expansion technique. We observe that these mathematical tools serve as tool for various engineering and medical designs.

1.0 Study Session Learning Outcomes

After you are done studying this session, you to be able to:

1. Apply the Taylor's and Maclaurin's series on functions with infinite series.
2.Solve up to n^{th} derivatives of any given function.
3. Use the Leibniz theorem in solving n^{th} derivatives of product of two functions.

2.0 Main Content

2.1 Successive Differentiation

If this is your first time studying this session you should first ask what is successive differentiation. This involves finding higher derivatives of a given function up to n[th] order. When a function $y = f\ (x)$ is differentiated with respect to x the differential coefficient is written as $\frac{dy}{dx}$ or $f'(x)$. If the expression is differentiated again, the second differential coefficient is obtained and is written as $\frac{d^2y}{dx^2}$ (pronounced dee two $y$by dee x squared) or $f''(x)$ (pronounced f double-dash x). By successive differentiation, further higher derivatives such as $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4} \dots \frac{d^ny}{dx^n}$ may be obtained.

Example 1: Find the n^{th} derivative of $y = ae^{bx}$ where $a, b\epsilon\mathbb{R}$

Solution

$$\frac{dy}{dx} = y' = abae^{bx}$$

The second derivative can be obtained as:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = y'' = ab^2ae^{bx}$$

The third derivative is found as:

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = y''' = ab^3ae^{bx}$$

By careful inspection, the n^{th} derivative can be deduced as:

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = y^n = ab^nae^{bx}$$

Example 2: Find the n^{th} derivative of $y = \sin(ax + b)$

Solution

$$y = \sin(ax + b)$$

$y' = a\cos(ax+b) = a\sin\left[(ax+b)+\frac{\pi}{2}\right]$ (by cofactor relationship)

$$y'' = a^2\cos\left[(ax+b)+\frac{\pi}{2}\right] = a^2\sin\left[(ax+b)+\frac{2\pi}{2}\right]$$

The third derivative can also be found as;

$$y''' = a^3\cos\left[(ax+b)+\frac{2\pi}{2}\right] = a^3\sin\left[(ax+b)+\frac{3\pi}{2}\right]$$

By careful inspection, the n^{th} derivative can be found as:

$$y^n = a^n\cos\left[(ax+b)+\frac{(n-1)\pi}{2}\right] = a^n\sin\left[(ax+b)+\frac{n\pi}{2}\right]$$

N.B: Similar procedure can be followed for $cos(ax+b)$.

A special case is obtained if $a=1, b=0$ in example 2, as;

$$\frac{d^n}{dx^n}(\sin x) = \sin\left[x+\frac{n\pi}{2}\right]$$

Example 3: Find then^{th} derivative of$y = \ln[(ax+b)(cx+d)]$

Solution

$y = \ln[(ax+b)(cx+d)] = \ln(ax+b)+\ln(cx+d)$ (applying the law of logarithm)

Next is to take the first derivative;

$$y' = \frac{a}{(ax+b)}+\frac{c}{(cx+d)} = a(ax+b)^{-1}+c(cx+d)^{-1}$$

Let us look at the second derivative,

$$y'' = a^2(-1)(ax+b)^{-2}+c^2(-1)(cx+d)^{-2}$$

You should follow the same procedure; the third derivative is obtained as:

$$y''' = a^3(-1)(-2)(ax+b)^{-3}+c^3(-1)(-2)(cx+d)^{-3}$$

For further clarity, the fourth derivative is computed as;

$$y'^v = a^4(-1)(-2)(-3)(ax+b)^{-4}+c^3(-1)(-2)(-3)(cx+d)^{-4}$$

So that by inspection, the n^{th} derivative can be generalized as;

$$y^n = a^n(-1)^{n-1}(n-1)!\,(ax+b)^{-n}+c^n(-1)^n(n-1)!\,(cx+d)^{-n}$$

Simplifying,

$$y^n = (-1)^n (n-1)! \left[\frac{a^n}{(ax+b)^n} + \frac{c^n}{(cx+d)^n} \right]$$

2.2 Leibnitz Theorem

The Leibnitz theorem helps to compute the n^{th} of product rule. It states that if f and g are n-times differentiable functions, then the product fg is also n-times differentiable and is give as:

$$[f(x)g(x)]^n = (fg)^n(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^k(x)$$

where

$\binom{n}{k} = nC_k = \frac{n!}{(n-k)!k!}$ is the binomial coefficient and $f^0(x) = f(x)$

So that we can write the formula out as:

$$[f(x)g(x)]^n$$
$$= [f(x)]^n g(x) + \binom{n}{1}[f(x)]^{n-1}[g(x)]' + \binom{n}{2}[f(x)]^{n-2}[g(x)]''$$
$$+ \binom{n}{3}[f(x)]^{n-3}[g(x)]''' + \cdots + \binom{n}{n-1}[f(x)]'[g(x)]^{n-1}$$
$$+ \binom{n}{n} f(x)[g(x)]^n$$

For us to have a clearer understanding of this formula, we should consider the following examples:

Example 4: Find the n^{th} derivative of $y = e^{ax} \sin bx$

Solution

Let $f = e^{ax}$ and $= \sin bx$, we know that the n^{th} derivative for each function are:

$f^n = a^n e^{ax}$, $\qquad g^n = b^n \sin\left(bx + \frac{n\pi}{2}\right)$

By the use of Leibnitz theorem,

$$(fg)^n = a^n e^{ax}\sin bx + na^{n-1}be^{ax}\sin\left(bx+\frac{\pi}{2}\right) + \frac{n(n-1)}{2!}a^{n-2}b^2e^{ax}\sin\left(bx+\frac{2\pi}{2}\right)+\cdots + nae^{ax}b^{n-1}\sin\left(bx+\frac{(n-1)\pi}{2}\right)+e^{ax}b^n\sin\left(bx+\frac{n\pi}{2}\right)$$

We can get special cases for $n = 1, n = 2$ and so on

Example 5: Use the Leibnitz theorem to find the n^{th} derivative of $y = \frac{ae^{nx}}{(3x-2)}$

Solution

Let $f = \boldsymbol{a}e^{bx}$ and $= (3x-2)^{-1}$, we know that the n^{th} derivative for each function are:

$f^n = ab^ne^{bx}$, $\qquad g^n = \frac{3^n(-1)^n n!}{(3x-2)^{n+1}}$

By the use of Leibnitz theorem,

$$(fg)^n = ab^ne^{bx}(3x-2)^{-1} - \frac{3nab^{n-1}e^{bx}}{(3x-2)^2} + \frac{3^2n(n-1)ab^{n-2}e^{bx}}{(3x-2)^3}+\cdots + nabe^{ax}\frac{3^{n-1}(-1)^{n-1}(n-1)!}{(3x-2)^n} + e^{ax}\frac{3^n(-1)^n n!}{(3x-2)^{n+1}}$$

In-text Question 1: *What is the* n^{th} derivative of $= \frac{1}{2x+1}$ *?*

Answer: $\frac{(-2)^n n!}{(2x+1)^{n+1}}$

2.3 Taylor's and Maclaurin's Series

Our concept of Taylor's series was formulated by the Scottish mathematician James Gregory and formally introduced in 1715 by the English mathematician Brook Taylor. If the Taylor series is cantered at zero, then that series is also called a **Maclaurin series**, named after the Scottish mathematician Colin Maclaurin, who made extensive use of this special case of Taylor series in the 18th century.

A function can be approximated by using a finite number of terms of its Taylor series. The polynomial formed by taking some initial terms of the Taylor series is called a Taylor polynomial. Your Taylor series of a function is the limit of that function's Taylor polynomials as the degree increases, provided that the limit exists. A function may not be equal to its Taylor series, even if its Taylor series converges at every point.

Taylor's series is considered to be the foundation of numerical analysis. It is used as a basic for many numerical formulas, Taylor'sseries is obtained from Taylor's theorem

Taylor's Theorem

If a continuous function $f(x)$ has a continuous $(n+1)th$ derivative on the interval $[x_0\,, x]$ then it can be represented by finite Taylor's series of the form

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!}f^n(x_0) + R_{n+1}(x)$$

where $R_{n+1}(x) = \frac{(x-x_0)^{n+1}}{(n+1)!}f^{n+1}(\varepsilon)x_0 \leq \varepsilon \leq x$

where $A.E = R_{n+1}(x)$

Maclaurin Series

Let us look at Maclaurin series.This is a special form of Taylor's series when the initial point is zero $(x_0 = 0)$

Substituting $x_0 = 0$ in Taylor series, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0) + AE = R_{n+1}(x)$$

$R_{n+1}(x) = \frac{x^{n+1}}{(n+1)!}f^{n+1}(\varepsilon)$ $\qquad x_0 \leq \varepsilon \leq x$

Maclaurin's series of a given function must satisfied the following condition:

1. $f(0) \neq \infty$
2. $f'(0), f''(0), f'''(0), \ldots \neq \infty$

3. The series must be convergent

Example 6:

Compute the fourth-degree polynomial approximation to the function $f(x) = \ln(1+2x)$ at(a) x_0= 2.0, (b) x_0= 0.0

Solution

(a) $f(x) = \ln(1+2x)$

You use Taylor's series because $x_0 \neq 0.;\ x_0 = 2.0,$

$$f(x_0) = f(2) = \ln(1+4) = \ln(5)$$

$$f'(x) = \frac{2}{1+2x}, f'(x_0) = f'(2) = \frac{2}{1+2x} = \frac{2}{5}$$

$$f''(x) = \frac{-1(2)(2)}{(1+2x)^2}, f''(2) = \frac{-4}{25}$$

$$f'''(x) = \frac{-1(-2)(2)(2)(2)}{(1+2x)^3} = \frac{16}{(1+2x)^3} = \frac{16}{125}$$

$$f'^{v}(x) = \frac{-1(-2)(-3)(2)(2)(2)(2)}{(1+2x)^4} = \frac{-96}{(1+2x)^4} = \frac{-96}{625}$$

Substituting into Taylor's Series

$$f(x) = \ln 5 + \frac{2(x-2)}{5} - \frac{4(x-2)^2}{50} + \frac{16(x-2)^3}{750} - \frac{96(x-2)^4}{15000}$$

(b) at x_0= 0.0 implies the Maclaurin's series;

$$f(x_0) = f(0) = \ln(1) = 0$$

$$f'(x) = \frac{2}{1+2x}, f'(x_0) = f'(0) = \frac{2}{1+2x} = 2$$

$$f''(x) = \frac{-1(2)(2)}{(1+2x)^2}, f''(0) = -4$$

$$f'''(x) = \frac{-1(-2)(2)(2)(2)}{(1+2x)^3} = \frac{16}{(1+2x)^3}, \qquad f'''(0) = 16$$

$$f'^{v}(x) = \frac{-1(-2)(-3)(2)(2)(2)(2)}{(1+2x)^4} = \frac{-96}{(1+2x)^4}, f'^{v}(0) = -96$$

Substituting into Taylor's Series

$$f(x) = 2x - 2x^2 + \frac{8x^3}{3} - 4x^4$$

Which is the required polynomial.

Example 7:

Compute the third-degree polynomial approximation to the function $f(x) = 4e^{x^2}$ at x_0= 1.0

Solution

We use Taylor's series because $x_0 \neq 0.;\ x_0 = 1.0,$

$$f(x_0) = f(1) = 4e^{1^2} = 4e$$

$$f'(x) = 8xe^{x^2}, f'(x_0) = f'(1) = 8e$$

$$f''(x) = 16x^2e^{x^2} + 8e^{x^2}, f''(1) = 24e$$

$$f'''(x) = 32x^3e^{x^2} + 32xe^{x^2} + 16xe^{x^2}, f'''(1) = 80e$$

Substituting into Taylor's Series

$$f(x) = 4e + 8e(x-1) + 12e(x-1)^2 + \frac{80e(x-1)^3}{6}$$

Which is the required polynomial.

In-text Question 2: *A Taylor's series at the point* $x_0 = 1.0$ *is called Maclaurin's series (TRUE/FALSE)*

Answer: *FALSE*

3.0Tutor Marked Assignments (Individual or Group)

1. Find then^{th} derivative of $y = \cos(ax + b)$

2. Find then^{th} derivative of $y = \frac{1}{1-5x+6x^2}$

3. If $y = \sin ax + \cos ax$. Show that $y^n = a^n[1 + (-1)^n \sin 2ax]^{1/2}$

4. Obtain a third degree polynomial approximation to

a. $f(x) = \sqrt{1+3x}$
b. $f(x) = Sinx$
c. $f(x) = e^{2x+1}$ about $x_0 = 0$, $x_0 = 0.5$

4.0 Conclusion/Summary

In this session, some basic mathematical methods such as successive derivatives, Taylor's and Maclaurin's series are developed. These series serve important role in linearization of non-linear term in mathematical modelling. It is expected that at the end of this session, you should be able to compute successive derivative of any differentiable function as well as the Taylor's series and Maclaurin series about a specific point.

In summary, we have established the successive derivative of any differentiable function and further went ahead to obtain the n-times derivative of product rule called the Leibnitz rule. Also, the Taylor's series was introduced as a tool for converting an infinite series to finite term by truncating after a given point. A special case of Taylor's series called Maclaurin's series is obtained when $x_0 = 0$

5.0 Self-Assessment Questions

1. Is the Taylor's series a special case of The Maclaurin's series?
2. What is the n^{th} derivative of the function $y = e^{mx} + x^m$

Answer to Self-Assessment Questions:

1. NO, the Maclaurin's series is a special case of the Taylor's series
2. $m^n e^{mx} + \frac{r!x^{m-n}}{(m-n)!}$

6.0 Additional Activities (Videos, Animations & Out of Class activities)

a. Visit U-tube http://bit.ly/2NXVKmM , http://bit.ly/2JySVV4 , http://bit.ly/2JOfnbR , http://bit.ly/30AnEGY , http://bit.ly/2XLMPth , http://bit.ly/2Ya4QRi. Watch the video & summarise in 1 paragraph

b. View the animation on successive derivatives, Leibniz theorem and critique it in the discussion forum

c. Take a walk and engage any 3 students on the difference between successive derivatives, Leibniz theorem; In 2 paragraphs summarise their opinion of the discussed topic.

7.0 References/Further Readings

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Study Session 3

Tangents, Normal and Radius of Curvature

Section and Subsection Headings:

Introduction:

Welcome to the third session of this course. I believe you have learned a lot so far. I encourage you to maintain the interest through the rest of the sessions.

In this session, we shall explore another mathematical method, namely; equation of tangents and normal and then further study the radius of curvature of a given curve. These concepts have significant application in oil exploration and engineering designs.

1.0 Study Session Learning Outcomes

After you are done studying this session, you should be able to:

1. Compute the radius of curvature of a given curve
2. Find the equation of tangents and normal about a curve
3. Differentiate between tangents and normal

2.0 Main Content

2.1Equation of Tangents and Normal

The equation of the tangent of a curve $y = f(x)$ at the point (x_1, y_1) is a straight line toucing the edge of the curve and is given as:

$$y - y_1 = m(x - x_1)$$

where m is the slope of curve at (x_1, y_1) and is defined as

$$m = \left.\frac{dy}{dx}\right|_{x_1}$$

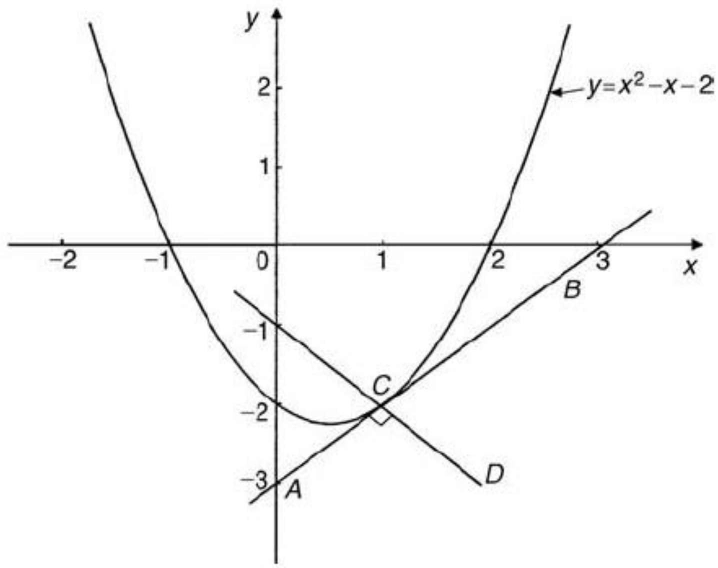

Figure 1.3.1: Pictorial form of tangent and normal

The normal at any point on a curve is the line which passes through the point and is at right angles to the tangent. Hence, in Fig. 28.12, the line $CD$is the normal.

It may be shown that if two lines are at right angles then the product of their gradients is -1. Thus if $m$is the gradient of the tangent, then the gradient of the normal is $-\frac{1}{m}$.

Hence the equation of the normal at the point $(x_1, y_1)$is given by:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Example 1

Find the equation of the tangent and normal to the curve $y = e^{-x} + 2x^2 - 1$ at the point $(1, -2)$.

Solution

First, we need to obtain the slope of tangent m

$m = \frac{dy}{dx} = -e^{-x} + 4x$ at $(1, -2)$.

$$m = 4 - e^{-1} = \frac{4e - 1}{e}$$

Now, our slope of normal is

$$m_2 = -\frac{1}{m} = \frac{e}{1 - 4e}$$

Since the slope of tangent as well as normal have been obtained;

The equation of tangent at $(1, -2)$ is;

$$y + 2 = \left(\frac{4e - 1}{e}\right)(x - 1)$$

$y = \left(\frac{4e-1}{e}\right)(x - 1) - 2$ is the equation of tangent of the curve $y = e^{-x} + 2x^2 - 1$ at the point $(1, -2)$.

The equation of normal is given by;

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y + 2 = \frac{e}{1 - 4e}(x - 1)$$

$y = \frac{e}{1-4e}(x - 1) - 2$ is the equation of normal to the curve $y = e^{-x} + 2x^2 - 1$ at the point $(1, -2)$.

Example 2: Find the equation of tangent and normal to the curve $P = \frac{1}{t}$ at the point $\left(3, \frac{1}{3}\right)$.

Solution

Slope of tangent is $m = \frac{dP}{dt} = \frac{-1}{t^2}$, at $t = 3$,

$$m = \frac{-1}{9}$$

So that the slope of normal is $m_2 = -\frac{1}{m} = 9$

The equation of tangent then becomes

$$P - \frac{1}{3} = -\frac{1}{9}(t - 3)$$

On simplifying, the equation of tangent becomes;

$$9P + t = 6$$

Similarly, the equation of normal is obtained as;

$P - \frac{1}{3} = 9(t - 3)$,

Which on further simplification yields;

$3P - 27t = 80.$

In-text Question 1: *Can the slope of tangent and normal ever be same?*

Answer: *NO, the slope of tangent and normal can never be the same*

2.2. Radius of Curvature

The radius of curvature has it relevance when discussing the surface tension of a fluid. For a curve, the radius of curvature equals the radius of the circular arc which best approximates the curve at that point and is defined as;

$$\rho = \left| \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \right|$$

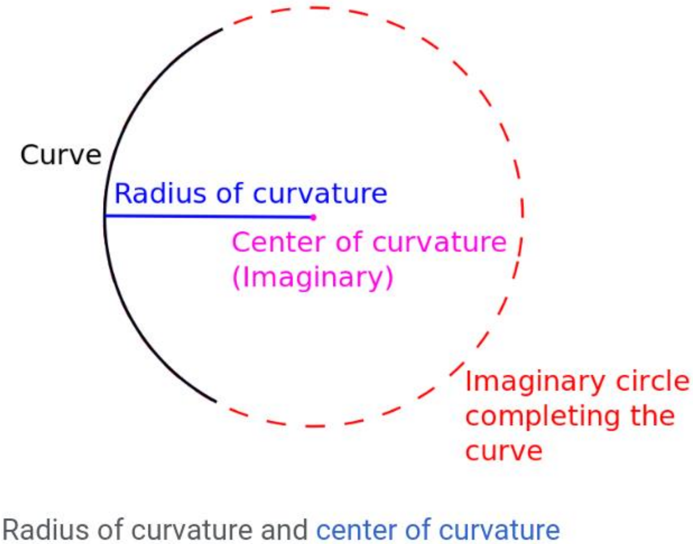

Figure 1.3.2: Radius of curvature and center of curvature

For parametric differentiation, the radius of curvature is given as;

Example 3:
Find the radius of curvature of the curve $y = 3x^2 + 3e^{2x}$

Solution

$$y = 3x^2 + 3e^{2x}$$

So that

$$y' = 6x + 6e^{2x} = 6(x + e^{2x})$$

Again, the second derivative;

$$y'' = 6 + 12e^{2x}$$

So that the radius of curvature is given as;

$$\rho = \left|\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right| = \rho = \left|\frac{[1 + \{6(x + e^{2x})\}^2]^{3/2}}{6 + 12e^{2x}}\right|$$

$$\rho = \left|\frac{[1 + 36x^2 + 12xe^{2x} + e^{4x}]^{3/2}}{6 + 12e^{2x}}\right|$$

Example 4: Find the radius of curvature of the part of surface having parametric equations; $x = 3t^2, y = \frac{3}{t}$ at $t = \frac{1}{2}$ to 4 decimal places.

Solution

$x = 3t^2, \frac{dx}{dt} = 6t$

$y = \frac{3}{t}, \frac{dy}{dt} = \frac{-3}{t^2},$

By parametric differentiation,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-3}{t^2}}{6t}$$

$\frac{dy}{dx} = \frac{-3}{6t^3}$ at $t = \frac{1}{2},$

$$\frac{dy}{dx} = -\frac{8}{2}$$

Next is to compute the second derivative;

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right] \times \frac{dt}{dx}$$

$$\frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{3}{2t^4}$$

$$\frac{d^2y}{dx^2} = \frac{3}{2t^4} \times \frac{1}{6t}$$

$\frac{d^2y}{dx^2} = \frac{1}{4t^5}$ at $t = \frac{1}{2}$

$$\frac{d^2y}{dx^2} = 8$$

Applying the formula,

$$\rho = \left| \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \right|$$

$$\rho = \left| \frac{\left[1 + \left(-\frac{8}{2}\right)^2\right]^{3/2}}{8} \right|$$

$$\rho = \left| \frac{\left[1 + \frac{64}{4}\right]^{3/2}}{8} \right| = \frac{[4 + 64]^{3/2}}{8 \times 8}$$

$$\rho = \frac{[68]^{3/2}}{64} 8.7616$$

In-text Question 2: *If the second derivative is zero, is it possible to compute the radius of curvature?(TRUE/FALSE)*

Answer: *FALSE (if the second derivative zero, we would have the radius of curvature as infinity)*

3.0Tutor Marked Assignments (Individual or Group)

1. Find the equation of tangent and normal to the curve $y = e^{-x} + 2x^2 - \tan x$ at the point $(1, 0)$.

2. Find the slope of tangent and normal to the curve $4y + x = 3e^{-xy} + 2x^2 - 1$ at the point $(1, 1)$.

3. Find the radius of curvature of the curve $3x^2y = 3x^2 \cos y + 3e^{2x}$ at (1,1)

4. Find the radius of curvature of the part of surface having parametric equations; $x = 3t^5, y = e^{2t}$ at $t = 1$.

4.0Conclusion/Summary

In this session, another mathematical method based on equation of tangent and normal as well as radius of curvature are presented. These mathematical formulae serve as a powerful tool in design of bridges and road networks. In addition, it is expected that every student should be able to find equation of tangent and normal to any given curve as well as computing the radius of curvature of a given function.

In summary, we established that the equation of tangent is a straight line drawn at the edge of a curve while the equation of normal is also a straight line at right angle to equation of normal. We also found that, if the slope of tangent is m, then the slope of normal is $-\frac{1}{m}$. So that the equation of tangent and normal at a point (x_1, y_1) are respectively given as: $y - y_1 = m(x - x_1)$ and $y - y_1 = -\frac{1}{m}(x - x_1)$. Further, for any given curve, the radius of curvature can be obtained as; $\rho = \left| \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \right|$.

5.0 Self-Assessment Questions

1. What is the radius of curvature of the curve $y = 3x^3 - 4x$ at $x = 0$?
2. Find the slope of the normal to the curve $y = e^x + 2x^2 - 1$ at (1,3)?

Answer to Self-Assessment Questions:

1. The radius of curvature is $\frac{\sqrt{(26)^3}}{18}$
2. The slope of normal is $-\frac{1}{4+e}$

5.0 Additional Activities (Videos, Animations & Out of Class activities)

a. Visit U-tube http://bit.ly/2xStlUA , http://bit.ly/30DgCRI , http://bit.ly/2SiOOzj , http://bit.ly/2XSvMpH , http://bit.ly/2Gg38no , http://bit.ly/2O0Z80j ,

http://bit.ly/2Ggklgp , http://bit.ly/2JN9U4O , http://bit.ly/2JDv0nH. Watch the video & summarise in 1 paragraph
b. View the animation on Tangents, Normal and radius of curvature and critique it in the discussion forum
c. Take a walk and engage any 3 students on the difference between Tangents and Normal and radius of curvature; In 2 paragraphs summarise their opinion of the discussed topic.

7.0References/Further Readings

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MODULE 2

Integral Calculus

Content

Study Session 1

Methods of Integration

Section and Subsection Headings:

Introduction:

You are welcome to a new study module, in previous sessions, we have so far learnt the applications of differential calculus which includes finding the equations of tangent and normal, successive derivative, Taylor's and Maclaurin's series, Leibnitz theorem and computation of radius of curvature.

In this this session, we shall explore and exemplify different techniques of integration. This forms the basis for solution of various mathematical modelling of engineering and medical applications. We shall also explore the technique of trigonometry identity, power formula, integration by substitution, by partial fraction and then integration by part.

1.0 Study Session Learning Outcomes

After you are done studying this session, you should be able to:
1.Compute the integral of any given integrable function
2.Solve integral involving product of two functions
3.Exemplify various methods of integration.

2.0 Main Content

2.1 Integral of basic functions

Let say when $y = x^n$, the derivative i.e. $\frac{dy}{dx} = nx^{n-1}$. If we differentiate $\frac{x^{n+1}}{n+1}$with respect to x.

That is $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = n + 1\frac{x^{n+1-1}}{n+1} = x^n$

Therefore, when $\frac{dy}{dx} = x^n$, then $y = \frac{x^{n+1}}{n+1}$, that is to say the integral of x^n with respect to x is $\frac{x^{n+1}}{n+1}$ (where $n \neq -1$).

The integral of a function is physically known as the area under a given curve, as shown in Figure 1.

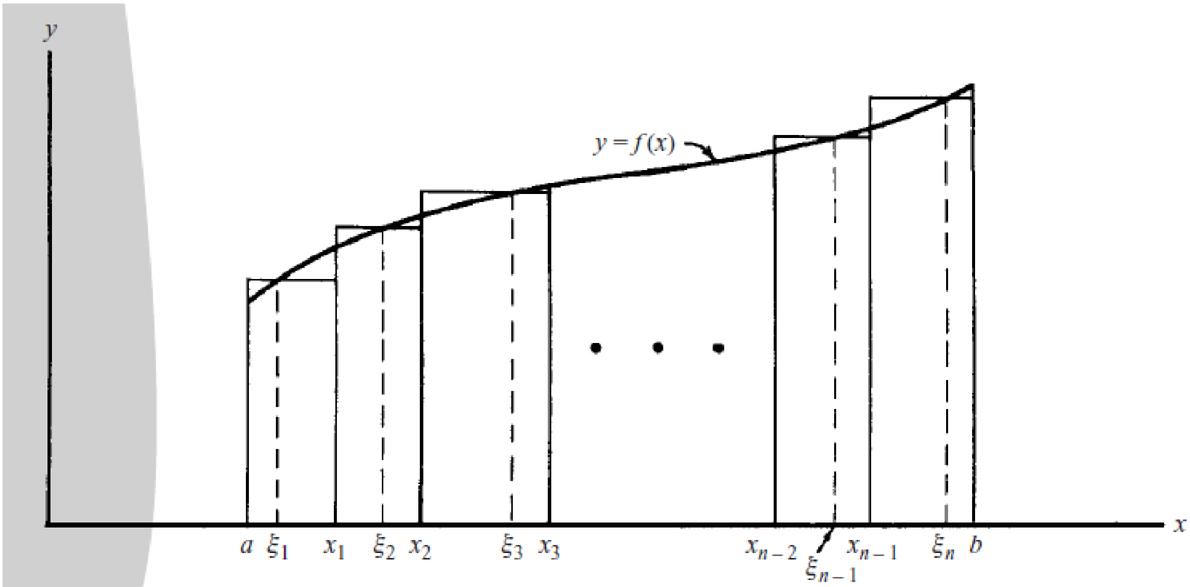

Figure 2.1.1: Schematic of integration

For further clarity on methods of integrations, we shall present the integral of basic elementary functions as a revision of Math105. Table 1 presents the integral of basic function for proper understanding when the techniques of integration shall be considered.

Table 2.1.1: Integration table for standard functions

	Function	Anti-derivative
1.	$y = x^n : n \neq -1, n \in \mathbb{R}$	$\frac{x^{n+1}}{n+1} + c$
2.	$y = sin\ ax : a \in \mathbb{R}$	$-\frac{cos\ ax}{a} + c$
3.	$y = cos\ ax : a \in \mathbb{R}$	$\frac{sin\ ax}{a} + c$
4.	$y = e^{ax} : a \in \mathbb{R}$	$\frac{e^{ax}}{a} + c$
5.	$y = log_e x = ln\ x$	$x(lnx - 1) + c$
6.	$y = \tan x$	$ln(secx) + c$
7.	$y = sec^2 x$	$\tan x + c$
8.	$y = \cot x$	$ln(sinx) + c$
9.	$y = f'(x)e^{f(x)}$	$e^{f(x)} + c$
10.	$\frac{1}{x}$	$\ln x + c$
11	$y = \int \frac{f'(x)}{f(x)} dx$	$\ln\lvert f(x)\rvert + c$
12	$y = a^x$	$\frac{a^x}{\ln a} + c$

2.1.2 Properties of integration

Another important note is the basic properties of integral functions;

If f and g are integrable functions, then:

1. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)\, dx + c$
2. $\int kf(x)dx = k \int f(x)dx + c : k \in \mathbb{R}$
3. $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)\, dx$
4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

6. $\int_a^a f(x)dx = 0$

It is important to state that a very important condition for integrability of a function in continuity over a closed and bounded interval.

2.2 Integration by substitution and trigonometric identity

Let $f(x)$ and $g(x)$ be integrable functions; the basic substitution techniques include:

(a) if the derivative of the denominator is equal to the numerator i.e.$y = \frac{f'(x)}{f(x)}$,

then $\int y\, dx = \ln|f(x)| + c$

(b) if $y = f(g(x))$ such that $g(x)$ is a linear function, then

$\int y\, dx = \int f(g(x))\, dx = \frac{\int f(u)}{g'(x)} du + c$ where $u = g(x)$

(c) if $y = f'(x)e^{f(x)}$,

then $\int y\, dx = \int f'(x)e^{f(x)}\, dx = e^{f(x)} + c$

Examples on integration by substitution and trigonometry identities

Example 1: find the anti-derivative of $y = (3x-1)^{100}$

Solution

It would be lengthened to expand the power up to 100. Since this look like a linear function in polynomial form, meaning we have to go by substitution

To find $\int (3x-1)^{100}\, dx$,

Let $u = (3x-1)$, then $\frac{du}{dx} = 3$, which means that $dx = \frac{du}{3}$.

Substituting the above transformation into the question,

$$= \frac{1}{3}\int u^{100}\, du$$

Using the power formula,

$$= \frac{u^{101}}{3 \times 101} + c$$

Substituting back the value of u,

$$\int (3x-1)^{100}\,dx = \frac{(3x-1)^{101}}{303} + c$$

Example 2: Evaluate $\int 3xe^{4x^2}\,dx$

Solution

Because the derive of the power of the exponential is linear, then we can apply the substitution method as follows:

Let $u = 4x^2$, then $\frac{du}{dx} = 8x$, which means that $dx = \frac{du}{8x}$.

Substituting the above transformation into the question,

$$= \int 3xe^u \frac{du}{8x}$$

$$= \frac{3}{8}\int e^u\,du$$

$$= \frac{3}{8}e^u + c$$

Substituting the value of u,

$$\int 3xe^{x^2}\,dx = \frac{3}{8}e^{4x^2} + c$$

Example 3: Evaluate $\int \frac{3x}{\sqrt{1-5x^2}}\,dx$

Solution

Let $u = \sqrt{1-5x^2}$, then $\frac{du}{dx} = -\frac{10x}{\sqrt{1-5x^2}}$, which means that $dx = -\frac{udu}{10x}$.

Substituting the above transformation into the question,

$$= -\int \frac{3x}{u}\frac{udu}{10x}$$

$$= -\frac{3}{10}\int du$$

$$= -\frac{3}{10}u + c$$

Substituting the value of u,

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{3}{10}\sqrt{1-x^2} + c$$

Example 4: Determine the integral $\int \frac{(1-t)^2}{\sqrt{t}}dt$

Solution

$$\int \frac{(1-t)^2}{\sqrt{t}}dt$$

By expansion, $(1-t)^2 = 1 - 2t + t^2$

$$= \int \frac{(1-2t+t^2)}{\sqrt{t}}dt$$

$$= \int \left(\frac{1}{\sqrt{t}} - \frac{2t}{\sqrt{t}} + \frac{t^2}{\sqrt{t}}\right)dt$$

$$= \int \frac{dt}{\sqrt{t}} - \int \frac{2t}{\sqrt{t}}dt + \int \frac{t^2}{\sqrt{t}}dt$$

By the use of Table 1,

$$\int \frac{(1-t)^2}{\sqrt{t}}dt = 2\sqrt{t} - \frac{4}{3}t^{3/2} + \frac{2}{5}t^{5/2} + c$$

Example 5: Solve $\int \frac{adx}{1+(bx)^2}$

Solution

Let $x = \frac{tan\theta}{b}$, implies, $\frac{dx}{d\theta} = \frac{sec^2\theta}{b}$, which means $dx = \frac{sec^2\theta d\theta}{b}$

Substituting the above transformation into the question,

$$\int \frac{adx}{1 + (bx)^2} = \int \frac{asec^2\theta d\theta}{b(1 + tan^2\theta)}$$

$= \frac{a}{b}\int d\theta$ (since $1 + tan^2\theta = sec^2\theta$)

$$= \frac{a}{b}\theta + c$$

But $x = \frac{tan\theta}{b}$ which implies that $bx = tan\theta$, therefore $tan^{-1}(bx) = \theta$

$$\int \frac{adx}{1 + (bx)^2} = \frac{atan^{-1}(bx)}{b} + c$$

Example 6: $\int \frac{dx}{\sqrt{a^2-x^2}}$

Solution

Let $x = asin\theta$, then $dx = acos\theta d\theta$,

Substituting the above transformation into the question,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{acos\theta d\theta}{\sqrt{a^2 - a^2sin^2\theta}}$$

$= \int \frac{acos\theta d\theta}{\sqrt{a^2(1-sin^2\theta)}}$ (but $(1 - sin^2\theta) = cos^2\theta$)

$$= \int \frac{acos\theta d\theta}{acos\theta}$$

$$= \int d\theta$$

$$= \theta + c$$

But $x = asin\theta$ which implies that $\frac{x}{a} = sin\theta$, therefore $sin^{-1}\left(\frac{x}{a}\right) = \theta$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = sin^{-1}\left(\frac{x}{a}\right) + c$$

2.3 Integration by Partial Fraction

In this case, the given equations comes of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions.We first have to resolve the function into their corresponding partial fraction form and then take the integral term by term:

Example 7: Evaluate $\int \frac{3dx}{(x-1)(x+2)(x-3)}$

Solution

Resolving the integral into their corresponding partial fraction form:

$$\frac{3}{(x-1)(x+2)(x-3)} = \frac{1}{5(x+2)} - \frac{1}{2(x-1)} + \frac{3}{10(x-3)}$$

So that

$$\int \frac{3dx}{(x-1)(x+2)(x-3)} = \int \left[\frac{1}{5(x+2)} - \frac{1}{2(x-1)} + \frac{3}{10(x-3)}\right] dx$$

$$= \int \frac{dx}{5(x+2)} - \int \frac{dx}{2(x-1)} + \int \frac{3dx}{10(x-3)}$$

$$= \frac{1}{5}\ln|x+2| - \frac{1}{2}\ln|x-1| + \frac{3}{10}\ln|x-3| + c$$

$$= \frac{1}{5}\ln|x+2| - \frac{1}{2}\ln|x-1| + \frac{3}{10}\ln|x-3| + c$$

Example 8: Determine $\int \frac{2xdx}{(x-1)(x+1)^2}$

Solution

Resolving the integral into their corresponding partial fraction form:

$$\frac{2x}{(x-1)(x+2)^2} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)} + \frac{1}{(x+1)^2}$$

So that

$$\int \frac{2xdx}{(x-1)(x+2)^2} = \int \left[\frac{1}{2(x-1)} - \frac{1}{2(x+1)} + \frac{1}{(x+1)^2}\right] dx$$

$$= \int \frac{dx}{2(x-1)} - \int \frac{dx}{2(x+1)} + \int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| - \frac{1}{x+1} + c$$
$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{x+1} + c$$

(using law of logarithm)

Example 9: Evaluate $\int \frac{(1+x^2)dx}{(x-1)(x+1)}$

Solution

Resolving the integral into their corresponding partial fraction form:

$\frac{(1+x^2)}{(x-1)(x+1)} = 1 + \frac{1}{(x-1)} - \frac{1}{(x+1)}$ (Integrating both sides)

So that

$$\int \frac{(1+x^2)dx}{(x-1)(x+1)} = \int dx + \int \frac{dx}{(x-1)} - \int \frac{dx}{(x+1)}$$

$= x + \ln|x-1| - \ln|x+1| + c$ (on simplifying)

$$\int \frac{(1+x^2)dx}{(x-1)(x+1)} = x + \ln\left|\frac{x-1}{x+1}\right| + c$$

In-text Question 1: *Every integrable function is continuous? (TRUE/FALSE)*
Answer: *True*

2.4 Integration by Parts and Reduction Formulae

2.4.1 *Integration by parts*

In this section, we would discuss integration of product of two or more different functions. From product rule,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to *x*;

$$uv = \int udv + \int vdu$$

So that the integration by part formula is given by:

$$\int udv = uv - \int vdu$$

Where one function is u and the other is dv

Example 10: Determine $\int xe^{2x}dx$

Solution

Let $u = x$, so that $dv = e^{2x}$

So that $du = dx$ and $v = \frac{e^{2x}}{2}$ (note that v is obtained by integrating dv)

Applying the integration by part formula;

$$\int udv = uv - \int vdu$$

$$\int xe^{2x}dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx$$

By integrating the remaining term, we have;

$$\int xe^{2x}dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$$

Note that integration is termed the removal of derivatives, hence, one to integrate until all derivatives are removed.

Example 11: Determine $\int xtan^{-1}xdx$

Solution

Let $u = tan^{-1}x$, so that $dv = x$

So that $du = \frac{1}{1+x^2}$ and $v = \frac{x^2}{2}$ (note that v is obtained by integrating dv)

Applying the integration by part formula;

$$\int udv = uv - \int vdu$$

$$\int xtan^{-1}xdx = \frac{x^2}{2}tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

By using partial fraction and trigonometric substitution,

$$\int xtan^{-1}xdx = \frac{x^2}{2}tan^{-1}x - \frac{1}{2}(x - tan^{-1}x) + c$$

Example 12: Determine $\int \ln x \, dx$

Solution

Let $u = \ln x$, so that $dv = dx$

So that $du = \frac{1}{x}$ and $v = x$ (note that v is obtained by integrating dv)

Applying the integration by part formula;

$$\int udv = uv - \int vdu$$

$$\int \ln x \, dx = x \ln x - \int \frac{x}{x} dx$$

$$\int \ln x \, dx = x \ln x - x + c$$

Example 13: Solve $\int \cos(ax)\, e^{bx} dx$

Solution

Let $Z = \int \cos(ax)\, e^{bx} dx$

Let $u = \cos(ax)$, so that $dv = e^{bx}$

So that $du = -asin(ax)$ and $v = \frac{e^{bx}}{b}$ (note that v is obtained by integrating dv)

Applying the integration by part formula;

$$\int udv = uv - \int vdu$$

$$Z = \frac{\cos(ax)\, e^{bx}}{b} + \frac{a}{b}\int \sin(ax)\, e^{bx} dx$$

Using integration by part again on $\int \sin(ax)\, e^{bx} dx$

Let $u = \sin(ax)$, so that $dv = e^{bx}$

So that $du = acos(ax)$ and $v = \frac{e^{bx}}{b}$ (note that v is obtained by integrating dv)

$$Z = \frac{\cos(ax)\, e^{bx}}{b} + \frac{a}{b}\left[\frac{\sin(ax)\, e^{bx}}{b} - \frac{a}{b}\int \cos(ax)\, e^{bx} dx\right]$$

$Z = \frac{\cos(ax) e^{bx}}{b} + \frac{a}{b}\left[\frac{\sin(ax) e^{bx}}{b} - \frac{a}{b} Z\right]$ (simplifying)

$$Z = \frac{b\cos(ax)\, e^{bx}}{a^2 + b^2} + \frac{a\sin(ax)\, e^{bx}}{a^2 + b^2}$$

2.4.2 Integration by Reduction Formulae

When using integration by parts as previously discussed, an integral such as $\int x^2 e^x dx$requires integration by parts twice. Similarly, $\int x^3 e^x dx$requires integration by parts three times. Thus, integrals such as $\int x^5 e^x dx$, $\int x^6 \sin 2x\, dx$and $\int x^2 \cos x\, dx$for example, would take a long time to determine using integration by parts. Reduction formulaeprovide a quicker method for determining such integrals and the method is demonstrated with the following examples:

Example 14: Find the reduction formulae for$\int x^n e^{ax}\, dx$. Hence, compute $\int x^2 e^{3x}\, dx$

Solution

By integration by parts;

Let $u = x^n$, $dv = e^{ax}$, $\frac{du}{dx} = nx^{n-1}$, $v = \frac{e^{ax}}{a}$

So that

$\int x^n e^{ax}\, dx = x^n \frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx$ (1)

Let $I_n = \int x^n e^{ax}\, dx$, so that

Using integration by part again

$$\int x^{n-1} e^{ax}\, dx = I_{n-1}$$

So that

$I_n = x^n \frac{e^{ax}}{a} - \frac{n}{a} I_{n-1}$ (2)

From equation (2), substitute $n = 2, a = 3$

$I_2 = x^2 \frac{e^{3x}}{3} - \frac{2}{3} I_1,$

$$I_1 = x\frac{e^{3x}}{3} - \frac{1}{3} I_0$$

$$I_0 = \int x^0 e^{3x}\, dx = \int e^{3x}\, dx = \frac{e^{3x}}{3}$$

Substituting the value of I_1, I_0, we have

$$I_2 = x^2 \frac{e^{3x}}{3} - \frac{2}{3}\left[x\frac{e^{3x}}{3} - \frac{1}{3}\left(\frac{e^{3x}}{3}\right)\right]$$

Simplifying,

$$I_2 = x^2 \frac{e^{3x}}{3} - \frac{2}{9} x e^{3x} - \frac{2}{27} e^{3x}$$

Example 15: Obtain the reduction formula for $\int x^n \sin ax\, dx$. Compute $\int x^2 \sin 5x\, dx$

Solution

By usual integration by parts (IBP),

Let $= x^n$, $du = nx^{n-1}$, $dv = \sin ax,\ v = -\frac{\cos ax}{a}$

Applying the IBP formula,

$\int x^n \sin ax\, dx = -\frac{x^n \cos ax}{a} + \frac{n}{a}\int x^{n-1} \cos ax\, dx$ (3)

If we let $I_n = \int x^n \sin ax\, dx$,

Applying IBP on $\int x^{n-1} \cos ax\, dx$

$u = x^{n-1}$, $du = (n-1)x^{n-2}$, $dv = \cos ax,\ v = \frac{\sin ax}{a}$

$\int x^{n-1} \cos ax\, dx = \frac{x^{n-1} \sin ax}{a} - \frac{(n-1)}{a}\int x^{n-2} \sin ax\, dx$ (4)

Substituting (4) into (3), we have;

$$I_n = -\frac{x^n \cos ax}{a} + \frac{n}{a}\frac{x^{n-1} \sin ax}{a} - \frac{n(n-1)}{a^2}\int x^{n-2} \sin ax\, dx$$

$I_n = -\frac{x^n \cos ax}{a} + \frac{n}{a}\frac{x^{n-1} \sin ax}{a} - \frac{n(n-1)}{a^2} I_{n-2} + c$ (5)

To compute $\int x^2 \sin 5x\, dx$, substitute $a = 5, n = 2$ in equation (5),

$I_2 = -\frac{x^2 \cos 5x}{5} + \frac{2}{5}\frac{x^1 \sin 5x}{5} - \frac{2}{25} I_0$ (6)

$I_0 = \int x^0 \sin 5x\, dx = \int \sin 5x\, dx = -\frac{\cos 5x}{5}$ (7)

Substitute (7) into (6), we have;

$I_2 = -\frac{x^2 \cos 5x}{5} + \frac{2}{5}\frac{x^1 \sin 5x}{5} + \frac{2 \cos 5x}{125} + c$

Therefore,

$$\int x^2 \sin 5x\, dx = -\frac{x^2 \cos 5x}{5} + \frac{2}{5}\frac{x^1 \sin 5x}{5} + \frac{2 \cos 5x}{125} + c$$

Example 16: Determine the reduction formula for $\int tan^n ax\, dx$

Solution

$\int tan^n ax\, dx = \int tan^{n-2}(ax) \tan^2(ax)\, dx$ (law of indices)

But $\tan^2(ax) = \sec^2 ax - 1$

$$I_n = \int tan^{n-2}(ax)(\sec^2 ax - 1)dx$$

$I_n = \int tan^{n-2}(ax) \sec^2 ax\, dx - I_{n-2}$ (8)

But substituting method,

$\int tan^{n-2}(ax) \sec^2 ax\, dx = \frac{tan^{n-1}(ax)}{a(n-1)} + c$ (9)

Substituting (9) into (8), we have;

$I_n = \frac{tan^{n-1}(ax)}{a(n-1)} - I_{n-2}$ (10)

Example 17: Derive the reduction formula for $\int (\ln x)^n dx$

Solution

Let $I_n = \int (\ln x)^n dx$

$u = (\ln x)^n$, $\quad du = \frac{n(\ln x)^{n-1}}{x}$, $\quad dv = dx$, $\quad v = x$

By IBP,

$$I_n = x(\ln x)^n - n\int (\ln x)^{n-1}$$

So that,

$$I_n = x(\ln x)^n - nI_{n-1}$$

In-text Question 2: *The integral of* $x^{2/100}$ *is*

Answer: $\frac{100}{102}x^{102/100} + c$

3.0 Tutor Marked Assignments (Individual or Group)

1. Solve $\int \sin{(bx)}\, e^{bx} dx$
2. $\int x^2 tan^{-1} x dx$
3. Solve $\int \frac{dx}{1+7x^2}$
4. Estimate $\int \frac{4dx}{(6x^2-x-2)}$
5. Evaluate $\int 4x^2 e^{x^3}\, dx$
6. Evaluate $\int 4xe^{2x}\, dx$
7. Evaluate $\int x^2 tan^{-1}(2x)\, dx$
8. Derive the reduction formulae of $\int (\ln x)^4 dx$
9. Obtain the reduction formula for $\int x^n \cos ax\, dx$. Compute $\int x^3 \cos 2x\, dx$

4.0 Conclusion/Summary

In this session, various techniques of integral calculus such as integration by substitution and trigonometry identity, integration by part and reduction formulae are presented. These integral techniques serve as powerful tool for solving engineering problems and mathematical models. The students are expected to be able to compute the integral of any integrable functions.

5.0Self-Assessment Questions

1. What is the integral of the function xe^{-x^2}
2. What is the integral of the function xe^{x}

Answer to Self-Assessment Questions

1. The integral is $-\frac{e^{-x^2}}{2} + c$
2. The integral is $(x-1)e^{x} + c$

5.0 Additional Activities (Videos, Animations & Out of Class activities)

a. Visit U-tubehttp://bit.ly/2XR9orz , http://bit.ly/32vroeN , http://bit.ly/2xSfDkm , http://bit.ly/2YYVDZw , http://bit.ly/2Y8cbB1 , http://bit.ly/2XRa14F , http://bit.ly/32skW8d , http://bit.ly/2Y4FjsJ , http://bit.ly/2LQUDTs , http://bit.ly/2JDnnNX , http://bit.ly/2XWhejI , http://bit.ly/32wz7co . http://bit.ly/2XRnLRS , http://bit.ly/2GhM40f , http://bit.ly/32xRihO , http://bit.ly/2LVEWuf , http://bit.ly/2JNb6VQ , http://bit.ly/2LVBxeL. Watch the video & summarise in 1 paragraph

b. View the animation on methods of integration and critique it in the discussion forum.

c. Take a walk and engage any 3 students on the difference between functions and relations; In 2 paragraphs summarise their opinion of the discussed topic.

7.0 References/Further Readings

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Study Session 2

Definite Integrals, Area Enclosed by a Plane Curve

Section and Subsection Headings:

Introduction:

It feels so interesting seeing you here again, in this session, we shall dive deeper into further discuss on applications of integral calculus which include; definite integral, and area under a plane curve. Once again welcome.

These concepts are of great importanceto us in calculating the area and volume of irregular shapes as well as centre of masses. It also has significant applications in understanding blood flood, drug administration and consumer surplus in economics. This knowledge forms the basis in obtaining approximate solution to highly non-linear differential equations which are basis for various technological and engineering designs.

1.0 Study Session Learning Outcomes

After you are done studying this session, you should be able to:

1. List the properties of definite integral.
2. Compute the definite integral of any integrable function.
3. Solve area enclosed under a plane curve.

2.0 Main Content

2.1 Definite Integral

Suppose $f(x)$ is integrable and $\int f(x)dx = F(x) + c$,

Then $\int_a^b f(x)dx = F(b) - F(a)$.

Integrals containing an arbitrary constant $c$in their results are called **indefinite integrals** since their precise value cannot be determined without further information. On the other hand, you can say, **definite integrals** are those in which limits are applied and hence does not contain any constant of integration. If an expression is written as $[x]_a^b$, 'b' is called the upper limit and 'a' the lower limit. The operation of applying the limits is defined as:$[x^2]_a^b = b^2 - a^2$.

For instance, $\int_1^2 x^4\, dx = \left[\frac{x^5}{5} + c\right]_1^2 = \left[\frac{2^5}{5} + c\right] - \left[\frac{1^5}{5} + c\right] = \frac{31}{5}$

It is important to note that the constant of integration will always cancel out.

Consider the area of the region bound by $y = f(x)$, the x –axis, and the joining vertical segments (ordinates) $x = a$ and $x = b$ as shown in Figure 1.

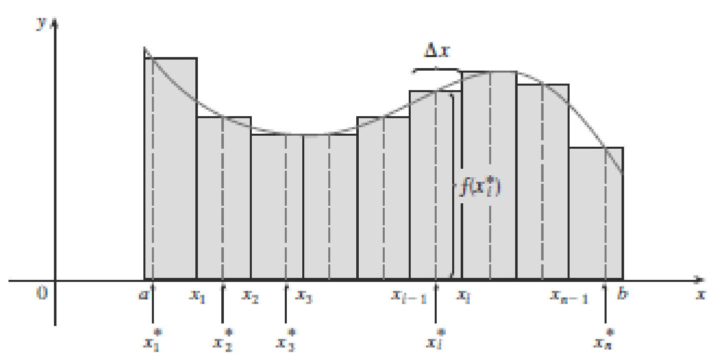

Figure 2.2.1: Pictorial representation of definite integral

Geometrically, this sum represents the total area of all rectangles in the above figure. Therefore, the integral is given as:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

This is called the definite integral of $f(x)$ between a and b.

Examples on definite integral

Example 1: Evaluate $\int_0^{\pi/2} \sin 2x \, dx$

Solution

The first approach is to evaluate the integral first using the any of the integration techniques discussed in previous section, so that:

$$\int_0^{\pi/2} sin\ 2x\, dx = -\frac{cos\ 2x}{2}\bigg|_0^{\pi/2}$$

$$= -\frac{cos\ 2(\pi/2)}{2} + \frac{cos\ 2(0)}{2}$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$= 0$$

Which implies that

$$\int_0^{\pi/2} sin\ 2x\, dx = 0.$$

Example 2: Compute $\int_0^1 e^{-4x}\, dx$

Solution

On integrating the function, we have:

$$\int_0^1 e^{-4x}\,dx = -\left[\frac{e^{-4x}}{4}\right]_0^1$$

$$\int_0^1 e^{-4x}\,dx = -\left[\frac{e^{-4}}{4} - \frac{e^{-0}}{4}\right]$$

$$\int_0^1 e^{-4x}\,dx = \left[\frac{1-e^{-4}}{4}\right]$$

Example 3: Compute $\int_1^3 \frac{2}{3x+1}\,dx$

Solution

On integrating the function, we have:

$$\int_1^3 \frac{2}{3x+1}\,dx = \left[\frac{2}{3}\ln(3x+1)\right]_1^3$$

$$\int_1^3 \frac{2}{3x+1}\,dx = \frac{2}{3}\ln 10 - \frac{2}{3}\ln 4$$

$$\int_1^3 \frac{2}{3x}\,dx = \frac{2}{3}\ln\frac{10}{4}$$

Example 4: Evaluate $\int_0^3 (x^4 + 3x - 6)\,dx$

Solution

$$\int_0^3 (x^4 + 3x - 6)\,dx = \left[\frac{x^5}{5} + \frac{3x^2}{2} - 6x\right]_0^3$$

$$\int_0^3 (x^4 + 3x - 6)\,dx = \left[\frac{3^5}{5} + \frac{3(3^2)}{2} - 6(3)\right] - \left[\frac{0^5}{5} + \frac{3(0^2)}{2} - 6(0)\right]$$

$$\int_1^3 (x^4 + 3x - 6)\,dx = \left[\frac{243}{5} + \frac{27}{2} - 18\right] - [0]$$

$$\int_1^3 (x^4 + 3x - 6)\,dx = \frac{441}{10}$$

In-text Question 1: *The integral of* $\int_0^{\pi} sinx\,dx$ is

Answer: 0

2.2 Area Enclosed by a Plane Curve

In this sub session, we shall discuss the concept of area enclosed by a plane curve. This concept forms the basis for computation of area of irregular shapes.

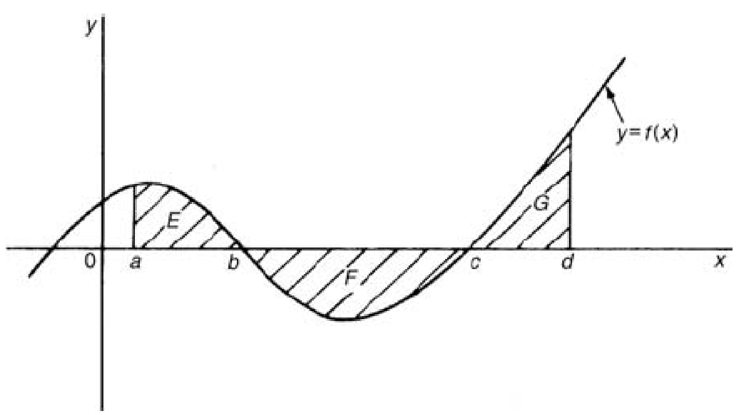

Figure 2.2.2: Physical representation of area under a curve

From Figure 2.2.2,

Total shaded area= $\int_a^b f(x)dx - \int_b^c f(x)dx + \int_c^d f(x)dx$

Example 5: Compute the area between the curves $y = x^3 - 2x^2 - 8x$ and the x −axis

Solution

By factorization,

$$x^3 - 2x^2 - 8x = x(x+2)(x-4)$$

So that the curve touches the x −axis at $x = 0, x = -2$ and $x = 4$ as shown in Figure 3 since the function is continuous.

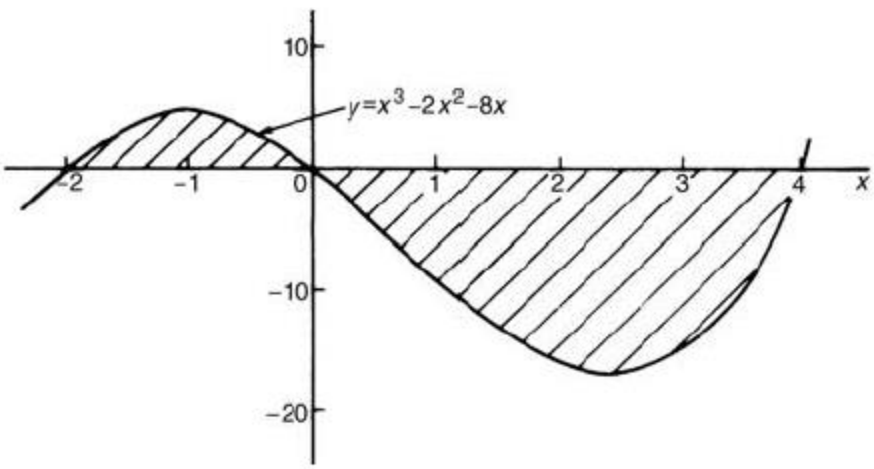

Figure 2.2.3: Graph of $y = x^3 - 2x^2 - 8x$ with the x −axis

So that the area of the shaded portion is

Total shaded area = $\int_{-2}^{0} y(x)dx - \int_0^4 y(x)dx$

$$= \int_{-2}^{0}(x^3 - 2x^2 - 8x)dx - \int_{0}^{4}(x^3 - 2x^2 - 8x)dx$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - 4x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - 4x^2\right]_{0}^{4}$$

$= 49\frac{1}{3}$ square units

Therefore, the area under the curve $y = x^3 - 2x^2 - 8x$ is $49\frac{1}{3}$ square units.

Example 6: Obtain the area enclosed between the curves $y = x^2 + 1$ and $y = 7 - x$.

Solution

The first thing you thing should do is to obtain the point of intersection between the two curves. At this point, the two curves are equal as follows:

$$x^2 + 1 = 7 - x$$

So that

$$x^2 + x - 6 = 0$$

By simple factorization,

$$\boldsymbol{(x - 2)(x + 3) = 0}$$

So that the two curves meet at $x = -3, x = 2$.

Obtaining the table of value and plotting for both curves on same plane as shown in Figure 2.2.4,

From the depicted graph, the area between the curves;

Shaded area $= \int_{-3}^{2}(7 - x)\,dx - \int_{-3}^{2}(x^2 + 1)\,dx$

$$= \int_{-3}^{2}(6 - x - x^2)\,dx$$

$$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \frac{22}{3} + \frac{27}{2}$$

$$= 20\frac{5}{6} \text{ square units}$$

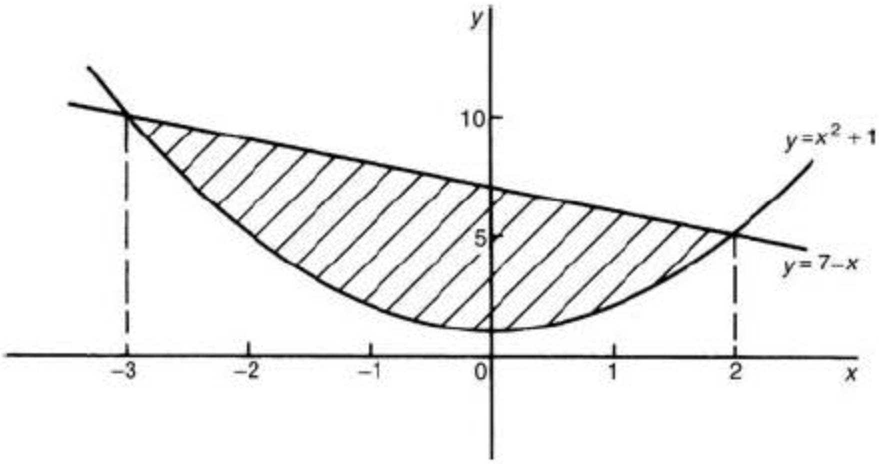

Figure 2.2.4: Portion between two curves

In-text Question 2: *Is the integral $\int_0^1 x^{1/2}\ dx$ equal to zero?*

Answer: *NO*

3.0 Tutor Marked Assignments (Individual or Group)

1. Compute:

(i) $\int_1^3 \frac{2}{3x^4} dx$

(ii) $\int_0^1 sec(3x)\, dx$

(iii) $\int_0^1 tan^{-1}(5x)\, dx$

(iv) $\int_0^3 (x^5 + 3x^7 - 100)\, dx$

(v) $\int_0^1 \left(x^{2/3} + 3x^{-4} - 6\right) dx$

(vi) $\int_0^3 xe^{-3x}\, dx$

(vii) Sketch the curves $y = x^2 + 3x$ and $y\ = 7 - 3x$ and determine the area enclosed by them.

(viii) Compute the area enclosed by the curve $y = 4cos(3x)$ at ordinates $x =$

0,and $x = \frac{\pi}{3}$.

4.0Conclusion/Summary

In this section, we presented the concept of definite integral as a special tool for finding area of shape under curve as well as area of irregular shapes between curves. Using basic integral techniques, we obtain the definite integral as: $\int f(x)dx = F(x) + c$, then $\int_a^b f(x)dx = F(b) - F(a)$. The concept of integral calculus can be summarised as: $\{\text{Differentiable functions}\} \subset \{\text{continuous functions}\} \subset \{\text{integrable functions}\}$.

5.0 Self-Assessment Questions

1. Evaluate the integral $\int_0^{\pi/2} \cos 2x\, dx$
2. Obtain the area enclosed by the curves $y = x^2 - 1$ and $y = x + 1$.

Answer to Self-Assessment Questions:

1. The value of the given integral is zero
2. The required area is $\frac{3}{2}$

6.0 Additional Activities (Videos, Animations & Out of Class activities)

a. Visit U-tubehttp://bit.ly/2JPFaQX , http://bit.ly/2GeipFm , http://bit.ly/2Y6eFQu , http://bit.ly/2XJItCQ , http://bit.ly/2xT2Uy1 , http://bit.ly/2XTJ7cr , http://bit.ly/32wfy44.

Watch the video & summarise in 1 paragraph

b. View the animation on definite integrals, area enclosed by a plane curve and critique it in the discussion forum

c. Take a walk and engage any 3 students on the difference between definite integrals and area enclosed by a plane curve; In 2 paragraphs summarise their opinion of the discussed topic.

7.0 References/Further Readings

Robert, C., Wrede and Murray, R., Spiegel, (2002), Advanced Calculus, 2nd Edition, Schaum's Outline Series McGRAW-HILL, New York Chicago San Francisco Lisbon London Madrid Mexico City Milan New Delhi San Juan Seoul Singapore Sydney Toronto.

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John Bird (2006), Higher Engineering Mathematics, 5th Edition, Linacre House, Jordan Hill, Oxford OX2 8DP 30 Corporate Drive, Suite 400, Burlington, MA01803, USA.

Study Session 3

Improper Integrals

Section and Subsection Headings:

Introduction:

You are welcome to study session 3, this session will give us the chance to explore the concept of improper integrals. This is well known that the requirement for integration of a function is continuity of that function.

We have cases where a function is not continuous at a point; such integrals are regarded as improper integrals. These integrals are classified into three kinds; first kind (infinite limits), second kind (unbounded interval) and third kind (combination of first and second kind). Different convergence tests a presented for better understanding of the concept

1.0 Study Session Learning Outcomes

After you are done studying this session, you should be able to:

1. Discuss the convergence of any improper integrals.
2. Differentiate between different types of improper integral.
3. Resolve different convergent test.

2.0 Main Content

2.1 Introduction to Improper Integrals

Every continuous function on a closed and bounded interval is integrable. But the converse is not always true. The integrals with above properties are called proper-integrals. When one more of the conditions is relaxed, the integrals are said to be improper. The integral $\int_a^b f(x)dx$ is called improper integral if:

(i) $a = -\infty$ or $b = \infty$ or both integration limits is infinite

(ii) $f(x)$ is unbounded at one or more points $a \leq x \leq b$. Such points are called singularities of $f(x)$ i.e. a point where $f(x)$is not defined

Integrals corresponding to (i) and (ii) are called improper integrals of the first and second kinds respectively. While integrals with both conditions (i) and (ii) are called improper integrals of third kind.

Examples

1. $\int_0^\infty cosx^2 dx$ is an improper integral of first kind [the integral is not finite]
2. $\int_0^5 \frac{dx}{x-1} dx$ is an improper integral of second kind [singularly unbounded at x=1]
3. $\int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx$ is an improper integral of third kind [both unbounded and singularly point at $x = 0$]
4. $\int_0^1 \frac{sinx}{x} dx$ is a proper integral since $\lim_{x\leftarrow 0^+} \frac{sinx}{x} = 1$

2.2 Improper Integrals of First Kind [Unbounded Intervals]

If f is an integrable function on the appropriate domains then the definite integrals $\int_a^x f(t)dt$ and $\int_x^a f(t)dt$ with variable upper and lower limits are function. Through them, we define three form of the improper integral of the first kind.

(a) If f is integrable on $a \leq x < \infty$, then $\int_a^\infty f(x)dx = \lim_{x\leftarrow\infty} \int_a^x f(t)dt$

(b) If f is integrable on $-\infty \leq x < a$, then $\int_{-\infty}^a f(x)dx$=$\lim_{x\leftarrow-\infty} \int_x^a f(t)dt$

(c) If f is integrable on $-\infty \leq x < a$, then

$\int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx = \lim_{x\leftarrow-\infty} \int_x^a f(t)dt$+ $\lim_{x\leftarrow\infty} \int_a^x f(t)dt$

2.2.1 Convergence or Divergence of Improper Integrals of First Kind

Let say if $f(x)$ be bounded and integrable in every finite interval $a \leq x \leq b$, then we define $\int_a^\infty f(x)dx$=$\lim_{b\leftarrow\infty} \int_a^x f(x)dx$where b is a variable on the positive real number.

If the limit exist, then $\int_a^\infty f(x)dx$ is convergent, otherwise, it is divergent.

Note:$\int_a^\infty f(x)dx$ bears close analogy to the infinite series $\sum_{n=1}^\infty u_n$

Similarly $\int_{-\infty}^b f(x)dx$=$\lim_{a\leftarrow-\infty} \int_a^b f(x)dx, a$ is a variable on negative real numbers also converges if $\lim_{a\leftarrow-\infty} \int_a^b f(x)dx$,exist and diverges otherwise.

Example 5: Discuss the convergence or divergence of $\int_1^\infty \frac{dx}{x^2}$

Solution

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\leftarrow\infty} \int_1^b \frac{dx}{x^2}$$

$$= \lim_{b\leftarrow\infty} \left(1 - \frac{1}{b}\right)$$

$$= 1$$

Since the limit exist and converges to 1, then the improper integral is convergent.

Example 6: Discuss the convergence or otherwise of $\int_{-\infty}^{u} cosxdx$

Solution

$\int_{-\infty}^{u} cosxdx = \lim_{a\leftarrow -\infty} \int_{a}^{u} cosxdx,$

$= lim_{a\leftarrow -\infty}[sinu - sina],$

since the limit does not exist, then the integral is divergent.

2.2.2 Special Improper Integrals of First Kind

1. Geometric or exponential integral $\int_{a}^{\infty} e^{-tx}\,dx$, where t is a constant, converges if $t > 0$ and diverges if $t \leq 0$

Note: $e^{-tx} = (e^{-tx})^{x} = r^{x}$, if we let $e^{-t} = r$

2. $p-$integral of the first kind $\int_{a}^{\infty} \frac{dx}{x^{p}}$, where p is a constant and $a > 0$, converges if $p > 0$ and diverges if $p \leq 1$

2.2.3 Convergence Test for Improper Integrals of First Kind

Let $f(x)$ be continuous $f(x)$is integrable on every finite interval $a \leq x \leq b$

1. Comparison test: for integrals with non-negative integrands

(a) **Convergence:** Let $g(x) \geq 0$ for all $x \geq a$, and suppose that $\int_{b}^{\infty} g(x)dx$ converges then if $0 \leq f(x) \leq g(x)$for all $x \geq a,$ then $\int_{a}^{\infty} f(x)dx$ also converges

Example 7: $\int_{0}^{\infty} \frac{dx}{e^{x}+4}$,

Since $\frac{1}{e^{x}+4} \leq \frac{1}{e^{x}} = e^{-x}$and $\int_{0}^{\infty} e^{-x}dx$converges [by exponential integral],

Hence,

$\int_{0}^{\infty} \frac{dx}{e^{x}+4}$ converges.

(b) **Divergence:** Let $g(x) \geq 0$for all $x \geq a$and suppose $\int_{a}^{\infty} g(x)dx$ diverges; then if $f(x) \geq g(x) \forall\, x \geq a,$ then $\int_{a}^{\infty} f(x)dx$ diverges.

Example 8: $\int_a^\infty \frac{dx}{lnx}$,

since $\frac{1}{lnx} > \frac{1}{x}$ for $x \geq 2$ and $\int_2^\infty \frac{dx}{x}$ diverges [by p −series]

therefore, $\int_2^\infty \frac{dx}{lnx}$ diverges

2. Quotient test

(a) If $f(x) \geq 0$ and $g(x) \geq a$ and if $\lim_{x\leftarrow\infty} \frac{f(x)}{g(x)} = A \neq 0$or ∞, then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ either both converges or both diverges.

(b) If $A = 0$ in (a), $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges

(c) If $A = \infty$in (a) and $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges

Theorem 1: $\lim_{x\leftarrow\infty} x^p f(x) = A$, then

(i) $\int_a^\infty f(x)dx$ converges if $p > 1$ and A is finite

(ii) $\int_a^\infty f(x)dx$ diverges if $p \leq 1$and $A \neq 0$

Example 9: $\int_0^\infty \frac{x^2dx}{4x^6+50}$converges, since $\lim_{x\leftarrow\infty} x^4 \frac{x^2}{4x^6+50} = \frac{1}{4}$

Example 10: $\int_0^\infty \frac{xdx}{\sqrt{8x^4+x^2+1}}$ diverges since $\lim_{x\leftarrow\infty} x.\frac{xdx}{\sqrt{8x^4+x^2+1}} = \frac{1}{8}$ but x is diverges by p- series

3. Series test: For integrals with non-negative integrands. $\int_a^\infty f(x)dx$ converges or diverges according as $\sum u_n$, where $u_n = f_n$ converges or diverges.

4. Absolute and conditional convergence: $\int_a^\infty f(x)dx$is called absolute convergence if $\int_a^\infty f(x)dx$ converges. If $\int_a^\infty f(x)dx$ converges but $\int_a^\infty |f(x)|dx$ diverges, then $\int_a^\infty f(x)dx$ is called conditionally convergent

Theorem 2: If $\int_a^\infty |f(x)|dx$ converges, then $\int_a^\infty f(x)dx$ converges, i.e. absolute Convergence Implies Integral Convergence

Example 11: $\int_0^\infty \frac{sinx}{x^2+1} dx$ is absolutely convergent, hence convergence since

$$\int_0^\infty |\frac{sinx}{x^2+1}|dx \leq \int_0^\infty \frac{dx}{x^2+1}$$

and $\int_0^\infty \frac{dx}{x^2+1}$ converges

because $\frac{1}{x^2+1} \leq \frac{1}{x^2}$ which is coverages by p −series

Example 12: $\int_0^\infty \frac{sinx}{x}$ is conditionally convergent but not absolute convergence.

Because, $\int_0^\infty |\frac{sinx}{x}| \leq \int_0^\infty \frac{1}{x}$ which diverges by p −test

In-text Question 1: *Does the integral* $\int_0^\infty \frac{sin2x}{3x}$ *converge?*

Answer: *YES*

2.3 Improper Integrals of Second Kind

If $f(x)$become unbounded any at the point $x = a$ of the interval $a \leq x \leq b$ then $\int_a^b f(x)dx$ =$\lim_{E\to0^+} \int_a^b f(x)dx$and we define it to be improper integral of the second series kind.Let say if the above limit exists, we call the integral on the left convergent; otherwise, it is divergent.

Similarly if $f(x)$ become unbounded only at the end $x = b$ of the interval $a \leq x \leq b$, then you extend the category of improper integrals of the second kind $\int_a^b f(x)dx$ =$\lim_{E\leftarrow0^+} \int_a^{b-E} f(x)dx$

Note that: unbounded is not the same as undefined

Example 13: $\int_0^1 \frac{sinx}{x} dx = \lim_{E\leftarrow0^+} \int_0^1 \frac{sinx}{x} dx$ is a proper integral,

since $\lim_{x\leftarrow0} \frac{sinx}{x} = 1$, hence is bounded as $x \to 0$, though the function is undefined at $x = 0$.

Finally, the category of improper integrals of the second kind also includes the case where $f(x)$ become unbounded only at an interior point $x = x_0$of the interval $a \leq x \leq b$, then you define,

$\int_a^b f(x)dx$ =$\lim_{E_1 \to 0^+} \int_a^{x_0+E} f(x)dx + \lim_{E_2 \to 0^+} \int_{x_0+E_2}^b f(x)dx$

(1)

The above interval converges or diverges if the above limit converges or diverges accordingly.

Cauchy principal value: It may happen that the limits on the right of (1) does not exist when $E_1 = E_2 = E$

$\int_a^b f(x)dx$ =$\lim_{E \to 0^+} \{ \int_a^{x_0+E} f(x)dx + \int_{x_0+E}^b f(x)dx\}$

(2)

If the limit on the right of (2) does exist, we call this limiting value the Cauchy principal value of the integral on the left.

$\int_0^x \frac{dt}{t}, = linx 0 < x < \infty$ is unbounded as $x \to 0$,this is an improper integral of the second kind also $\int_0^\infty \frac{dt}{t}$is of third $\int_1^\infty \frac{dt}{t}$ is of first kind which also diverges by p −integral.

2.3.1 Special improper integrals of the second kind

1. $\int_a^b \frac{dx}{(x-a)^p}$converges if $p < 1$and diverges if $p \geq 1$
2. $\int_a^b \frac{dx}{(b-x)^p}$converges if $p < 1$and diverges if $p \geq 1$

This can be called p integrals of the second kind. Note that when $p \leq 0$the integrals are proper.

2.3.2 Convergence test for improper integrals of the second kind

The following tests are given for the case where $f(x)$ is unbounded only at $x = a$ in the interval $a \leq x \leq b$

1. ***Comparison test for integrals with non-negative integrands***

(a) **Convergence:** Let $g(x) \geq 0$ and $0 \leq f(x) \leq g(x)$for $a \leq x \leq b$ and suppose that $\int_a^b g(x)dx$ converges, then $f(x)$ converges.

Example 14: Verify the convergence of $\int_1^5 \frac{dx}{\sqrt{x^4-1}}$

Solution

Since $\frac{dx}{\sqrt{x^4-1}} < \frac{dx}{\sqrt{x-1}}$ for $x > 1$,

then $\int_1^5 \frac{dx}{\sqrt{x-1}}$ converges (p integral with a=1, $p = \frac{1}{2}$)

hence $\int_1^5 \frac{dx}{\sqrt{x^4-1}}$ also converges

(b) **Divergence:** let $g(x) \geq 0$ for $a \leq x \leq b$, and suppose that $\int_a^b g(x)dx$ diverges. Then if $f(x) \geq g(x)$ for $a < x \leq b$ $\int_a^b f(x)dx$ also diverges.

Example 15: Verify the convergence or divergence of $\int_3^6 \frac{Inx}{(x-3)^4}$

Solution

$\int_3^6 \frac{Inx}{(x-3)^4} > \int_3^6 \frac{dx}{(x-3)^4}$diverges ($p$ integral with $a = 3, p = 4$)

Then $\int_3^6 \frac{Inx}{(x-3)^4} dx$also diverges $\frac{Inx}{(x-3)^4} > \frac{1}{(x-3)^4}$

2. Quotient Test for Integrals with Non-Negative Integrands

(a) if $f(x) \geq 0$and $g(x) \geq 0 a < x \leq b$ and if $\lim_{x \leftarrow a} \frac{f(x)}{g(x)} = A \neq 0$ or ∞ then$\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ either both converge or both diverge.

(b) If $A = 0$ in (a) and $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges

(c) If $A = \infty$ in (a) and $\int_a^b g(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges

Theorem: let $\lim_{x - a^+} (x-a)^p f(x) = A$

(i) $\int_a^b f(x)dx$ converges if $p < 1$and A is finite

(ii) $\int_a^b f(x)dx$ diverges if $p \geq 1$and $B \neq 0(B$ may be infinite)

Example 16:$\int_1^5 \frac{dx}{\sqrt{x^4-1}}$ converges,

since $\lim_{x-1^+}(x-1)^{1/2}\frac{1}{(x^4-1)^{1/2}} = \lim_{x-1^+}\sqrt{\frac{x-1}{x^4-1}} = \frac{1}{2}$

Example 17:$\int_6^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$ diverges

since $\lim_{x-3^-}(3-1).\frac{dx}{(3-x)\sqrt{x^2+1}} = \frac{1}{\sqrt{10}}$

$\int_a^b f(x)dx$ is called absolute convergent if $\int_a^b |f(x)|dx$ converge. If $\int_a^b f(x)dx$ converges but $\int_a^b |f(x)|dx$ diverges, then $\int_a^b f(x)dx$ is called conditionally convergent

Theorem 3: If $\int_a^b |f(x)|dx$ converge, then $\int_a^b f(x)dx$ converges absolute

Example 18: Since $\left|\frac{sinx}{\sqrt[3]{x-\pi}}\right| \leq \frac{1}{\sqrt[3]{x-\pi}}$ and $\int_\pi^{4\pi} |\frac{sinx}{\sqrt[3]{x-\pi}}|dx$converges and thus $\int_\pi^{4\pi}\frac{sinx}{\sqrt[3]{x-\pi}}dx$converges (absolutely)

improper integrals of the third kind can be expressed in terms of improper integrals of the first and second kinds, and hence the question of their convergence or divergence is answer by using results already established.

Example 18

1. Classify the following to the type of improper integral.

(a) $\int_{-1}^1 \frac{dx}{\sqrt[3]{x(x+1)}}$ (2^{nd} kind because its unbounded at $x = -1$)

(b)$\int_0^\infty \frac{dx}{1+tanx}$ 3^{rd} kind [integration limit is infinite and its unbounded when tan $tanx = -1$]

(c) $\int_{3}^{10} \frac{x^2}{(x-2)^{10}} dx$ proper integral

(d) $\int_{-\infty}^{\infty} \frac{x^2}{x^4 + x^4 + 1} dx$(first kind, No value of x for which $x^4 + x^4 + 1$=0)

(e)$\int_{1}^{\pi} \frac{1-cosx}{x^2} dx$ [proper integral [because using L'Hopital rule, $\lim_{x-1^+} \frac{1-cosx}{x^2} = \frac{1}{2}$]

Example 19: Transform $\int_{1}^{2} \frac{dx}{\sqrt{x(2-x)}}$into:

(a) an improper integral of first kind

(b) a proper integral

Solution

(a) Consider $\int_{1}^{2-E} \frac{dx}{\sqrt{x(2-x)}}$ where $0 < \in < 1,\ say$;let $2 - x = \frac{1}{y}$then we have $\int_{1}^{1/\in} \frac{dy}{y\sqrt{2y-1}}$ as $\in -0^+$is equivalent to $\int_{1}^{\infty} \frac{dy}{y\sqrt{2y-1}}$ which is improper integral of first kind.

(b) Let$2 - x = v^2$ in integral of a; it become $2\int_{\sqrt{\in}}^{1} \frac{dv}{\sqrt{v^2+2}}$ which can be considered as $2\int_{0}^{1} \frac{dv}{\sqrt{v^2+2}}$ which is a proper integral.

In-text Question 2: *Is the improper integral* $\int_{1}^{\infty} \frac{xdx}{6x^5+4x^2+1}$ *convergent?*

Answer: *YES*

3.0Tutor Marked Assignments (Individual or Group)

Examine the convergence of

1. (a). $\int_{1}^{\infty} \frac{Inx}{x+a} dx$ (b) $\int_{0}^{\infty} \frac{1-cosx}{x^2} dx$ (c) $\int_{\infty}^{-1} \frac{e^x}{x} dx$ (d) $\int_{-\infty}^{\infty} \frac{x^2+x^2}{x^6+1} dx$
2. (a) prove that $\int_{1}^{\infty} \frac{cosx}{x^2} dx$ converges

   (b) prove that $\int_{0}^{\infty} \frac{sinx}{x} dx$ converges, is it absolutely convergent?

3. (a) prove that $\int_{-1}^{7} \frac{dx}{(x-1)^3}$ converges (i) in the usual sense

(ii) in the Cauchy principal value sense

4. Investigate the convergence of

(a) $\int_{2}^{3} \frac{dx}{x^2(x^3-8)^{2/3}}$ Converges (d) $\int_{-1}^{1} \frac{2^{\sin^{-1}x}}{1-x} dx$ diverges

(b) $\int_{0}^{\pi} \frac{\sin x}{x^3} dx$ Diverges (e) $\int_{0}^{/2} \frac{dx}{(\cos x)^{1/n}} n > 1$ converges

(c) $\int_{1}^{5} \frac{dx}{\sqrt{(5-x)(x-1)}}$ Converges

5. If n is a real number, prove that $\int_{0}^{\infty} x^{n-1} e^{-x} dx$ (a) converges if $n > 1$

(b). Diverges of $n > 1$

(ii) Show that $\int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

(iii) Evaluate $\int_{0}^{\infty} e^{-x^2} \cos \alpha x dx$

4.0 Conclusion/Summary

In this session, the concept of improper integral is discussed in full detail. They arise when a definite integral has unbounded interval or it is discontinuous at a point in between the interval. The improper integrals are divided into 3 parts (first kind, second kind and third kind). Difference test of convergence for the three kinds of improper integrals are presented. Improper integrals of first kind arises when one or both limits are infinity. Improper integral of second kind on the other hand occurs if the functionis unbounded at one or more points $a \leq x \leq b$. At the end of this session, you should be able to differentiate between integrals of first, second or third kinds.

5.0 Self-Assessment Questions

1. What kind of convergence does the improper integral $\int_{0}^{\infty} \frac{\sin x}{x^2+1} dx$ satisfy?
2. When is the improper integral $\int_{0}^{\infty} f(x)\, dx$ convergent?

Answer to Self-Assessment Question:

1. Absolutely convergent
2. If $\lim_{a\to\infty} \int_0^a f(x)\, dx$ exists

5.0 Additional Activities (Videos, Animations & Out of Class activities)

a. Visit U-tubehttp://bit.ly/2xT4Q9L , http://bit.ly/2JLQuNR , http://bit.ly/2Gcn4HB , http://bit.ly/30EJjOy.Watch the video & summarise in 1 paragraph

b. View the animation on Improper integrals and critique it in the discussion forum

c. Take a walk and engage any 3 students on the concept ofImproper integrals; In 2 paragraphs summarise their opinion of the discussed topic.

8.0 Self Assessment Question Answers

for $x \geq 2$,

$\frac{x^2-1}{\sqrt{x^6+1}} \geq \frac{1}{2x}$,

since $\frac{1}{2}\int_2^\infty \frac{dx}{x}$ diverges

Hence $\int_2^\infty \frac{x^2-1}{\sqrt{x^6+16}}dx$ diverges

9.0 References/Further Readings

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MODULE 2

Partial Differentiation

Content

Study Session 1

Partial Derivative and Jacobians

Section and Subsection Headings:

Introduction:

Welcome once again, in this session, we shall be discussing the concept of partial derivatives.

This kind of derivative is used when the function considered has more than one independent variable. In this case, the derivative is carried out with respect to one of the variables and other kept constant. Also, Jacobian, as a powerful tool in solving system of equations and method of variation of parameters. We are made to understand that these concepts have significant application in mathematical modelling.

1.0 Study Session Learning Outcomes

After you are done studying this session, you should be able to:

1.Obtain the partial derivative of any function.

2.Solve the Jacobian of a given function of two or more variables.

3.State the theorems of Jacobian.

2.0 Main Content

2.1 Functions of Two or More Variables

The distinction for functions of two or more variables is that the domain is a set of $n-$tuples of numbers. The range remains one dimensional and is referred to an interval of numbers. If $n = 2$, the domain is pictured as a two-dimensional region. The region is referred to a rectangular Cartesian coordinate system described through number pairs (x, y), and the range variable is usually denoted by z. The domain variables are independent while the range variable is dependent.

We use the notation $f(x, y), F(x, y)$, etc., to denote the value of the function at $(x, y)$and write $z = f(x, y)$, $z = F(x, y)$, etc. We shall also use the notation $z = z(x, y)$although it should be understood that in this case z is used in two senses, namely as a function and as a variable.

Example 1: If $f(x, y) = x^2 + 2y^3$, obtain $f(3, -1)$

Solution

Then $f(3, -1) = (3)^2 + 2(-1)^3 = 7$

The concept is easily extended. Thus $w = f(x, y, z)$ denotes the value at (x, y, z) [a point in three dimensional space], etc.

Example 2: Compute $f(1,2,3)$ given $f(x, y, z) = x^2 + 2z^3y - z$

Solution

$$f(1,2,3) = (1)^2 + 2(3)^3 2 - 3$$
$$f(1,2,3) = 106$$

Example 3: If $z = \sqrt{1 - (x^2 + y^2)}$, the domain for which z is real consists of the set points (x, y) such that $x^2 + y^2 \leq 1$, i.e the set of points inside and on the circle in the xy plane having center at $(0,0)$ and radius 1.

In-text Question 1: *If a function $f(x)$ has n number of independent variables, then $f(x)$ is n-dimensional (TRUE/FALSE)*

Answer: *TRUE*

2.2 Partial Derivatives

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant, is called the partial derivative of the function with respect to the variable. Partial derivatives of $f(x, y)$with respect to x *and* y are denoted respectively by:

$$\frac{\partial f}{\partial x}\left[or\ f_x, f_x(x, y), \left.\frac{\partial f}{\partial x}\right|_y\right]$$

and

$$\frac{\partial f}{\partial y}\left[or\ f_y, f_y(x, y), \left.\frac{\partial f}{\partial x}\right|_x\right]$$

the latter notations being used when it is needed to emphasize which variables are held constant.

By definition, the partial derivative is given as:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x},$$
$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

when these limits exist, the derivatives evaluated at the particular point (x_0, y_0) are often indicated by $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0, y_0)$ *and* $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0, y_0)$respectively.

Example 4

If $f(x,y) = 2x^3 + 3xy^2$,

Then$f_x = \frac{\partial f}{\partial x}$

$= 6x^2 + 3y^2$

and

$$f_y = \frac{\partial f}{\partial y}$$

$= 6xy.$

Also $f_x(1,2) = 6(1)^2 + 3(2)^2$

$= 18,$

$f_y(1,2) = 6(1)(2) = 12.$

If a function f has continuous partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ in a region, then f must be continuous in the region. However, the existence of these partial derivatives alone is not enough to guarantee the continuity of f at that point.

Example 5: If $f(x,y) = 2x^2 - xy + y^2$, find (a)$\frac{\partial f}{\partial x}$, (b)$\frac{\partial f}{\partial y}at(x_0, y_0)$directly from the definition.

Solution

(a) $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$

$$= \lim_{h\to 0} \frac{[2(x_0+h)^2 - (x_0+h)y_0 + {y_0}^2] - [2{x_0}^2 - x_0 y_0 + {y_0}^2]}{h}$$

$$= \lim_{h\to 0} \frac{4hx_0 + 2h^2 - hy_0}{h}$$
$$= \lim_{h\to 0} 4x_0 + 2h - y_0$$

Applying limit $h \to 0$,

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = 4x_0 - y_0$$

(b) $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}$

$$= \lim_{k\to 0} \frac{[2{x_0}^2 - x_0(y_0 + k) + (y_0 + k)^2] - [2{x_0}^2 - x_0y_0 + {y_0}^2]}{k}$$
$$= \lim_{k\to 0} \frac{-kx_0 + 2ky_0 + k^2}{k}$$
$$= \lim_{k\to 0}(-x_0 + 2y_0 + k)$$

Applying limit $k \to 0$,

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = -x_0 + 2y_0$$

Since the limits exist for all points(x_0, y_0), we can write$f_x(x, y) = f_x = 4x - y, f_y(x, y) = f_y = -x + 2y$, which are functions of x *and* y.

Note that formally, $f_x(x_0, y_0)$ is obtained from $f(x, y)$ by differentiating f with respect to x, keeping y constant and then putting $x = x_0, y = y_0$. Similarly, $f_y(x_0, y_0)$ is obtained by differentiating f with respect to y, keeping x constant.

In-text Question 2: *Compute* $\theta_x(1,1)$ *if* $\theta(x, y) = \sin(3xy)$

Answer: $-\frac{cos\,(3)}{3}$

2.2.1 Higher Order Partial Derivatives

If $f(x,y)$ has partial derivatives at each point (x,y) in a region, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are themselves functions of x and y, which may also have partial derivatives. These second derivatives are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx},$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{xx},$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx},$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$ and the order of differentiation is unimportant; otherwise they may not be equal.

Example 6:

If $f(x,y) = 2x^3 + 3xy^2$,compute (a)f_{xx} (b)f_{yy} (c) f_{xy}(d) f_{yx} at (1,2)

Solution

$f_x = 6x^2 + 3y^2,$

$f_{xx} = 12x,$

$f_y = 6xy,$ so that

$f_{yy} = 6x,$

$f_{xy} = 6y$

$f_{yx} = 6y.$

Since $f_{xy} = f_{yx},$

Hence the function is continuous everywhere.

So that

$$f_{xx}(1,2) = 12,$$
$$f_{yy}(1,2) = 6,$$
$$f_{xy}(1,2) = f_{yx}(1,2) = 12.$$

In a similar manner, higher order derivatives are defined. For example $\frac{\partial^3 f}{\partial x^2 \partial y} = f_{yxx}$ is the derivative of f taken once with respect to y and twice with respect to x.

Example 7: If $z = x^2 \tan^{-1}\frac{y}{x}$, find $\frac{\partial^2 z}{\partial x \partial y}$ at $(1,1)$.

$$\frac{\partial z}{\partial y} = x^2 \frac{1}{1 + (y/x)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$
$$= x^2 \frac{x^2}{x^2 + y^2}\frac{1}{x}$$
$$= \frac{x^3}{x^2 + y^2}$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{x^3}{x^2 + y^2}\right)$$
$$= \frac{(x^2 + y^2)(3x^2) - (x^3)(2x)}{(x^2 + y^2)^2}$$
$$= \frac{2\,.\,3 - 1\,.\,2}{2^2} = 1 \;\; at \;\; (1,1)$$

The result can be written $z_{xy}(1,1) = 1$

2.2.2 Differentiation of Composite Functions (Chain Rule)

Let $z = f(x, y)$ where $x = g(r, s)$, $y = h(r, s)$ so that z is a composite function of $r \; and \; s$. Then;

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r},$$
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

In general, if $u = F(x_1, \dots, x_n)$where$x_1 = f_1(r_1, \dots r_p), \dots, x_n = f_n(r_1, \dots, r_p)$, then,

$$\frac{\partial u}{\partial r_k} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial r_k} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial r_k} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial r_k} \qquad k = 1,2, \dots, p$$

If in particular $x_1, x_2, \dots, x_n$ depend on only one variable s, then

$$\frac{du}{ds} = \frac{\partial u}{\partial x_1}\frac{dx_1}{ds} + \frac{\partial u}{\partial x_2}\frac{dx_2}{ds} + \cdots + \frac{\partial u}{\partial x_n}\frac{dx_n}{ds}$$

These results, often called chain rules, are useful in transforming derivatives from one set of variables to another. Higher derivatives are obtained by repeated application of the chain rules.

Example 8: If$z = f(x, y)$and $x = \emptyset(t),\ y = \varphi(t)$where $f, \emptyset, \varphi$ are assumed differentiable, Prove $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

Solution

We have by definition,

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0} \left[\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}\right]$$
$$= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Since as $\Delta t \to 0$ we have $\Delta x \to 0,\ \Delta y \to 0, \epsilon_1 \to 0, \epsilon_2 \to 0, \frac{\Delta x}{\Delta t} \to \frac{dx}{dt},\ \frac{\Delta y}{\Delta t} \to \frac{dy}{dt}$.

So that $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

Example 9: If $z = e^{xy^2},\ x = tcost,\ y = tsint,$ compute $\frac{dz}{dt}\ at\ t = \pi/2.$

S
olution

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \left(y^2 e^{xy^2}\right)(-tsint + cost) + \left(2xye^{xy^2}\right)(tcost + sint)$$

At $t = \pi/2\,,\ x = 0, y = \pi/2.$

Then $\frac{dz}{dt}\,\Big|_{t=\pi/2} = (\pi^2/4)(-\pi/2) + (0)(1)$

$$= -\frac{\pi^2}{8}$$

Example 10: Let $z = f(x,y)$ where $x = \emptyset(u,v),\ y = \varphi(u,v),$ prove that

$$(a)\,\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\,(b)\,\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

Solution

(a) Assuming that $f, \emptyset, \varphi,$ are differentiable, we have;

$$\frac{dz}{du} = \lim_{\Delta u \to 0}\frac{\Delta z}{\Delta u} = \lim_{\Delta u \to 0}\{\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta u} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta u} + \epsilon_1\frac{\Delta x}{\Delta u} + \epsilon_2\frac{\Delta y}{\Delta u}\}$$

$$= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

(b) The result is proved as in (a) by replacing $\Delta u\ by\ \Delta v\ \ and\ letting\ \Delta v \to 0.$

2.3 Jacobians

If $F(u,v)\ and\ \ G(u,v)$ are differentiable in a region, the *Jacobian determinant*, or briefly the *Jacobian*, of $F\ and\ G$ with respect to $u\ and\ v$ is the second order functional determinant defined by

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

Similarly, the third order determinant

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

is called the Jacobian of $F, G\ and\ H$ with respect to $u, v\ and\ w$.

2.2.3 Partial Derivatives Using Jacobians

Jacobians often prove useful in obtaining partial derivatives of implicit functions. Thus, for example, given the simultaneous equations

$$F(x,y,u,v) = 0 \qquad G(x,y,u,v) = 0$$

we may, in general, consider u and v as functions of x and y

$$\frac{\partial u}{\partial x} = \frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

$$\frac{\partial u}{\partial y} = \frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

$$\frac{\partial v}{\partial x} = \frac{\frac{\partial(F,G)}{\partial(x,u)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

$$\frac{\partial v}{\partial y} = \frac{\frac{\partial(F,G)}{\partial(y,u)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

The ideas are easily extended. Thus, if we consider the simultaneous equations

$$F(u,v,w,x,y) = 0 \qquad G(u,v,w,x,y) = 0 \qquad H(u,v,w,x,y) = 0$$

You may, for example, consider $u, v, and\ w$ as functions of $x\ and\ y$. In this case,

$$\frac{\partial u}{\partial x} = \frac{\frac{\partial(F,G,H)}{\partial(x,u,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}$$

$$\frac{\partial u}{\partial y} = \frac{\frac{\partial(F,G,H)}{\partial(y,u,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}$$

with similar results for the remaining partial derivatives.

2.3.4 Theorems on Jacobians

In the following, we assume that all functions are continuously differentiable.

1. A necessary and sufficient condition that the equations $F(u, v, x, y, z) = 0$, $G(u, v, x, y, z) = 0$ can be solved for $u\ and\ v$ (for example) is that $\frac{\partial(F,G)}{\partial(u,v)}$ is not identically zero in a region $\mathcal{R}$. Similar results are valid for m equations in n variables, where $m < n$.

2. If $x\ and\ y$ are functions of $u\ and\ v$ while $u\ and\ v$ are functions of $r\ and\ s$, then

$$\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(r,s)}$$

This is an example of a chain rule for *Jacobians*. These ideas are capable of generalization.

3. If $u = f(x, y)\ and\ \ v = g(x, y)$, then a necessary and sufficient condition that a functional relation of the form $\emptyset(u, v) = 0$ exists between $u\ and\ v$ is that $\frac{\partial(u,v)}{\partial(x,y)}$ be identically zero. Similar results hold for n functions of n variables.

Example 11: If $F(u, v, w, x, y) = 0,\ G(u, v, w, x, y) = 0, H(u, v, w, x, y) = 0$, find (a) $\left.\frac{\partial v}{\partial y}\right|_x$, (b) $\left.\frac{\partial x}{\partial v}\right|_w$, (c) $\left.\frac{\partial w}{\partial u}\right|_y$

Solution

Form 3 equations in 5 variables, we can (theoretically at least) determine 3 variables in terms of the remaining 2. Thus, 3 variables are dependent and 2 are independent. If we were asked to determine $\partial v/\partial y$, we would know that v is a dependent variable and y is an independent variable, but would not know the remaining independent variable. However, the particular notation $\left.\frac{\partial v}{\partial y}\right|_x$ serves to indicate that we are to obtain $\partial v/\partial y$ keeping x constant, i.e., x is the other independent variable.

(a) Differentiating the given equations with respect to y, keeping x constant, gives

$$(1) F_u u_y + F_v v_y + F_w w_y + F_y = 0$$
$$(2) G_u u_y + G_v v_y + G_w w_y + G_y = 0$$
$$(3) H_u u_y + H_v v_y + H_w w_y + H_y = 0$$

Solving simultaneously for v_y, we have

$$v_y = \left.\frac{\partial v}{\partial y}\right|_x = -\frac{\begin{vmatrix} F_u & F_y & F_w \\ G_u & G_y & G_w \\ H_u & H_y & H_w \end{vmatrix}}{\begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}}$$
$$= -\frac{\frac{\partial(F,G,H)}{\partial(u,y,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}$$

Equations (1), (2), and (3) can also be obtained by using differentials. The Jacobian method is very suggestive for writing results immediately, as seen in this problem. Thus, observe that in calculating $\frac{\partial v}{\partial y}\Big|_x$ the result is the negative of the quotient of two Jacobians, the numerator containing the independent variable y, the denominator containing the dependent variable v in the same relative positions. Using this scheme, we have

(b) $\left.\frac{\partial x}{\partial v}\right|_w = -\frac{\frac{\partial(F,G,H)}{\partial(v,y,u)}}{\frac{\partial(F,G,H)}{\partial(x,y,u)}}$

(c) $\left.\frac{\partial w}{\partial u}\right|_y = -\frac{\frac{\partial(F,G,H)}{\partial(u,x,v)}}{\frac{\partial(F,G,H)}{\partial(w,x,v)}}$

3.0Tutor Marked Assignments (Individual or Group)

1. If $z = e^{2xy^3},\ x = t^2 cost,\ y = tsin2t,$ compute$\frac{dz}{dt}\ at\ \ t = 0$.
2. If $f(x,y) = \ 2x^2y + 3x^3y^2$,compute (a)$f_{xxx}$ (b)f_{yy} (c) f_{xyx}(d) f_{yx} at (1,0)

4.0Conclusion/Summary

The concept of first order and higher order partial derivatives are introduced in this session. These concepts serves as fundamental in obtaining solutions to various physical partial differential equations. In addition, the Jacobian otherwise known as Jacobian determinant is introduced at the later part of this session. It is expected that you should be able to perform partial derivative of order one and two.

5.0Self-Assessment Questions

1. Differential the function $f(x,y,z) = x^2 \cos 2y + xz$ with respect to x.
2. If $z = \sin xy$, $x = x(t)$ and $y = y(t)$, find an expression for $\frac{dz}{dt}$.

Answer to Self-Assessment Questions:

1. $\frac{\partial f}{\partial x} = 2x \cos 2y + z$
2. $\frac{dz}{dt} = \cos xy \left[y\frac{dx}{dt} + x\frac{dy}{dt}\right]$

6.0 Additional Activities (Videos, Animations & Out of Class activities)

a. Visit U-tube http://bit.ly/2LpBg4y , http://bit.ly/2Y9yshM , http://bit.ly/2Y9I0cA , http://bit.ly/2LpgIZE , http://bit.ly/32upBXj ,

http://bit.ly/30BFidq , http://bit.ly/32x9sQL. Watch the video & summarise in 1 paragraph
b. View the animation on https://www.youtube.com/watch?v=Uz0MtFlLD-k and critique it in the discussion forum
c. Take a walk and engage any 3 students on the concepts of Partial derivatives and Jacobians; In 2 paragraphs summarise their opinion of the discussed topic.

7.0References/Further Readings

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Study Session 2

Lagrange's Multiplier Differentials and Linear Approximations

Section and Subsection Headings:

Introduction:

You are welcome to study session 2, this will be our last study module of this course.Lagrange's multiplier, minimum and maximum of a function of two or more variables will be discussed extensively.
These concepts have a vital application in optimisation and operations research. It is hoped that you will be able to apply this knowledge to solve physical problem in engineering designs and business administration.

1.0 Study Session Learning Outcomes

After you are done studying this session, you should be able to:
1. Compute the minimum, maximum and saddle points.
2. Resolve problems using Lagrange's multiplier.

2.0 Main Content

2.1 Lagrange's Undetermined Multipliers

Closely allied to the problem of locating the stationary points of some function $u = f(x, y)$ is the problem of locating points where $u = f(x, y)$ attains its greatest or its least value (an extremal value) subject to the condition that $x \ and \ y$ are related to each other via the equation

$$\emptyset(x, y) = 0$$

The problem can be clarified if we consider it graphically.

The graph of $u = f(x, y)$ is a surface within the (x, y, u) coordinate system Figure 1. Selecting a plane parallel to the $x - y$ plane on which the value of u is constant, u_k, we see that the surface intersects the plane in a curve given by the equation $f(x, y) = u_k$

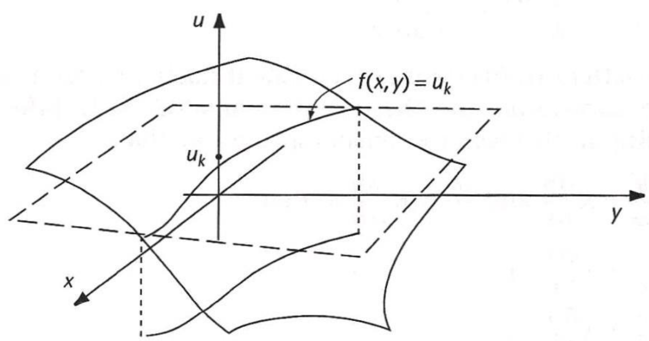

Figure 3.2.1: Pictorial representation of Lagrange's multiplier

The line of intersection can now be projected onto the $x - y$ plane to form what is known as a level curve. Different values of u_k determine different planes (all parallel to the $x - y$ plane), different level curves. Accordingly, an alternative graphical description of $u = f(x, y)$ is that of a family of level curves in the $x - y$ plane with each member of the family being associated with a particular value of u_k, where we assume $u_1 < u_2 < u_3 < \cdots < u_n \ \ or \ u_1 > u_2 > u_3 > \cdots > u_n$(Figure 2).

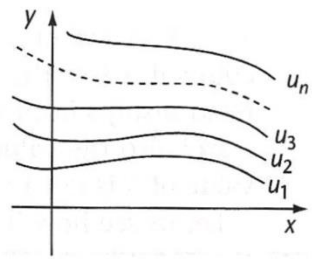

Figure 3.2.2: Representation of Lagrange's multiplier

We now superimpose onto this family of level curves the graph of the constraint equation (Figure 3.2.2);

$$\emptyset(x, y) = 0$$

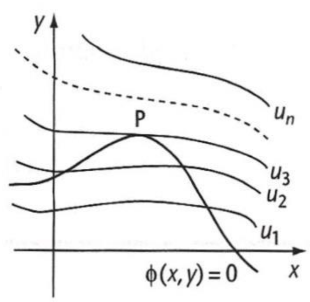

Figure 3.2.3: Graph of the constraint equation

Clearly, in the Figure 3 above, u_3 is the extremal value of $f(x, y)$ that coincides with $\emptyset(x, y) = 0$, and at the point P where they meet they share the same tangent line $\frac{dy}{dx}$. Now since $\emptyset(x, y) = 0$ $we\ see\ that$

$$\frac{dy}{dx} = -\frac{\partial\emptyset/\partial x}{\partial\emptyset/\partial y}$$

Because

$$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy = 0$$

so that $\frac{dy}{dx} = -\frac{\partial\emptyset/\partial x}{\partial\emptyset/\partial y}$

The same tangent can be found from

$$du = \frac{\partial f}{\partial x}dx = \frac{\partial f}{\partial y}dy$$

By equating the differential $du = 0$. Therefore

$$\frac{dy}{dx} = -\frac{\partial\emptyset/\partial x}{\partial\emptyset/\partial y} = -\frac{\partial\emptyset/\partial x}{\partial\emptyset/\partial y}$$

The latter two fractions are equivalent which means that the two numerators and the two denominators each differ by the same multiplicative factor k enabling us to say that,

$$\frac{\partial f}{\partial x} = k\frac{\partial\emptyset}{\partial x} \quad and \quad \frac{\partial f}{\partial y} = k\frac{\partial\emptyset}{\partial y}$$

So that

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial\emptyset}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial\emptyset}{\partial y} = 0 \qquad (2)$$

$\lambda = -k$ is called a Lagrange's multiplier and equations (1) and (2), coupled with the constraint equation $\emptyset(x, y) = 0$, gives us three relationships from which the values of $x \; and \; y$ at the extremal points – and also the value of λ if required – can be found. Quite often the value of λ is not important.

Example 1

Find the stationary points of the function $u = x^2 + y^2$ subject to the constraint $x^2 + y^2 + 2x - 2y + 1 = 0$

In this case, $u = x^2 + y^2$

$$\emptyset = x^2 + y^2 + 2x - 2y + 1$$

$$\frac{\partial u}{\partial x} = 2x;$$

$$\frac{\partial u}{\partial x} = 2y;$$

$$\frac{\partial\emptyset}{\partial x} = 2x + 2;$$

$$\frac{\partial\emptyset}{\partial x} = 2y - 2$$

Then we form and solve

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial\emptyset}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \lambda\frac{\partial\emptyset}{\partial y} = 0$$

together with

$$\emptyset = x^2 + y^2 + 2x - 2y + 1 = 0$$

Which gives

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial\emptyset}{\partial x} = 0$$
$$\therefore 2x + \lambda(2x + 2) = 0$$
$$\therefore x + \lambda(x + 1) = 0$$
$$\frac{\partial u}{\partial y} + \lambda\frac{\partial\emptyset}{\partial y} = 0$$
$$\therefore 2y + \lambda(2x - 2) = 0$$
$$\therefore y + \lambda(y - 1) = 0$$
$$\therefore \quad \frac{x}{y} = \frac{-\lambda(x + 1)}{-\lambda(y - 1)}$$
$$\therefore \quad xy - x = xy + y$$
$$\therefore y = -x$$

Substituting this in $\emptyset$

$$x^2 + x^2 + 2x + 2x + 1 = 0$$
$$2x^2 + 4x + 1 = 0$$
$$\therefore \quad x = -1 \pm \frac{\sqrt{2}}{2}$$
$$\therefore \quad y = 1 \mp \frac{\sqrt{2}}{2}$$

But $y = -x$

To find λ, we have $x + \lambda(x + 1) = 0 \qquad \therefore \; \lambda = \sqrt{2} \mp 1.$

In-text Question 1: *If a function is a constant, then its stationary point is zero (TRUE/FALSE)*

Answer: *TRUE*

2.2. Differentials

Let $\Delta x = dx \ and \ \Delta y = dy$ be increments given to $x \ and \ y. \ Then$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = \Delta f$$

is called the increment in $z = f(x, y)$. If $f(x, y)$ has continuous first partial derivatives in a region, then

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial x}\Delta y + \epsilon_2 \Delta x + \epsilon_2 \Delta y$$
$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial x}dy + \epsilon_1 dx + \epsilon_2 dy$$
$$= \Delta f$$

where $\epsilon_1 \ and \ \epsilon_2$ approach zero $\Delta x \ and \ \Delta y$ approach zero. The expression

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial x}dy$$
$$or \ df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dy$$

Is called the total differential or simply differential of $z \ or \ f,$ or the principal part of $\Delta z \ or \ \Delta f.$

2.2.1 Theorems on Differentials

1. If $z = f(x_1, x_2, \dots, x_n)$, then

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n$$

Regardless of whether the variables $x_1, x_2, \dots, x_n$ are independent or dependent on other variables.

2. If $f(x_1, x_2, \dots, x_n) = c,$ a constant, then $df = 0$. Note that in this case $x_1, x_2, \dots, x_n$ cannot all be independent variables.

3. The expression $P(x,y)dx + Q(x,y)dy$ or briefly $Pdx + Qdy$ is the differential of $f(x,y)$ if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. In such case $Pdx + Qdy$ is called an exact differential.

   Note: Observe that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ implies that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.
4. The expression $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ or briefly $Pdx + Qdy + Rdz$ is the differential of $f(x,y,z)$ if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$. In such case $Pdx + Qdy + Rdz$ is called an exact differential.

2.3. Maxima, Minima and Saddle Points for Functions of Two Variable

In this sub session, we shall give a step by step procedure on how to compute the minimum, maximum or saddle point of functions of two variables.

Procedure:

Given $z = f(x,y)$

(i) Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(ii) For stationary points, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

(iii) Solve the simultaneous equations $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial yx} = 0$ for x and y, which gives the coordinates of the stationary points.

(iv) Determine $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$

(v) For each of the coordinates of the stationary points, substitute value of x and y $into$ $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ and evaluate each.

(vi) Evaluate $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ for each stationary point.

(vii) Substitute the values of $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ into the equation

$$D = (\frac{\partial^2 z}{\partial x \partial y})^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) \ \ i.e \ \ (F_{xy})^2 - F_{xx}.F_{yy}$$

If:

(a) $D > 0,$ then the stationary point is a saddle point.

(b) $D < 0 \ \ and \ \ \frac{\partial^2 z}{\partial x^2} < 0,$ then the stationary point is a maximum point.

(c) $D < 0 \ \ and \ \ \frac{\partial^2 z}{\partial y^2} > 0,$ then the stationary point is a minimum point.

Example 2:

Find the stationary points of the surface $f(x,y) = x^3 - 6xy + y^3$ and determine its nature.

Solution

Let $z = f(x,y) = x^3 - 6xy + y^3$

$$f_x = 3x^2 - 6y,$$

$$f_{xx} = 6x$$

$$f_y = -6x + 3y^2,$$

$$f_{yy} = 6y$$

$$f_{xy} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = -6$$

For stationary points;

$$3x^2 - 6y = 0$$

$$-6x + 3y^2 = 0$$

Solving the above equations simultaneously

$$6y = 3x^2$$

$$\Rightarrow y = \frac{x^2}{2};$$

Substituting

$$\Rightarrow x = 0, 2$$

$$\text{when} \ \ x = 0, \ \ y = 0$$

when $x = 2, \ y = 2$

Thus, the stationary points occur at $(0,0) \ and \ (2,2)$

Now we have to test the nature of these stationary points;

For $(0,0)$; $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} = 0$ and

$$\frac{\partial^2 z}{\partial x \partial y} = -6$$

$$\Rightarrow \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 36$$

For $(2,2)$; $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} = 12$ and

$$\frac{\partial^2 z}{\partial x \partial y} = -6$$

$$\Rightarrow \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 36$$

$D(0,0) = 36 - 0 = 36$ (saddle point)

Since $D(0,0) = 36 > 0$, then the stationary points (0,0) is a saddle point

$D(2,2) = 36 - 144 = -108$ (minimum point)

On the other hand, $D(2,2) = -108 < 0$, hence the stationary point (2,2) is a saddle point.

Example 3: Determine the stationary values of $z = 5xy - 4x^2 - y^2 - 2x - y + 5$

Solution

Firstly, we shall compute the stationary points;

$$\frac{\partial z}{\partial x} = 5y - 8x - 2;$$

$$\frac{\partial z}{\partial y} = 5x - 2y - 1$$

Solving the two criticl point simultaneously,

$$\therefore \quad 8x - 5y + 2 = 0$$
$$5x - 2y - 1 = 0$$

gives $x = 1,\ y = 2$

Thus, the only stationary value occurs at $(1,2)$

Now substitute values of x and y $into$ $(\frac{\partial^2 z}{\partial x \partial y})^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)$

Therefore $\frac{\partial^2 z}{\partial x^2} = -8;$

$$\frac{\partial^2 z}{\partial y^2} = -2;$$
$$\frac{\partial^2 z}{\partial x \partial y} = 5$$

$$\therefore \quad \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = (5)^2 - (-8)(-2) = 9 \quad i.e \ > 0$$

Since $D(1,2) = 9 > 0$, therefore the stationary point at$(1,2)$is a saddle point.

In-text Question 2: *The stationary point of* $f(x) = 4x - 3$ *is*

Answer: 4

3.0Tutor Marked Assignments (Individual or Group)

1. Determine the stationary values of $z = 5x^2y - 4x^2 - y^2 - 2x - 4y + 50$
2. Find the stationary points of the surface $f(x, y) = 3x^3 - 6x^2y + y^3$ and determine its nature.
3. Find the stationary points of the function $u = 3x^2 + y^2$ subject to the constraint $x^2 + 4y^2 - 2xy - 2y + 100 = 0$

4.0 Conclusion/Summary

In this session, we discussed the concept of Lagrange's multiplier as well as minimum, maximum and saddle point of functions of two variables. This concept is applicable in engineering design, business management and optimization. You are expected to be able to discuss the nature of the stationary point of functions of two or more variables.

5.0 Self-Assessment Question

Find the stationary points of the function $z = x^2y - xy$.

Answer to Self-Assessment Question:

$(0.5,\ 0),\ x = 0,\ x = 1.$

5.0 Additional Activities (Videos, Animations & Out of Class activities)

a. Visit U-tubehttp://bit.ly/2GaLyRN , http://bit.ly/2LpC5ua , http://bit.ly/2NZYYGG , http://bit.ly/2XXsD33 , http://bit.ly/30Bfj60. Watch the video & summarise in 1 paragraph

b. View the animation on Lagrange's multiplier, differentials and linear approximation and critique it in the discussion forum

c. Take a walk and engage any 3 students on the difference between Lagrange's multiplier, differentials and linear approximation; In 2 paragraphs summarise their opinion of the discussed topic.

7.0 References/Further Readings

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Glossary

absolute value
a real number regardless of its sign
angle
the space between two lines or planes that intersect
arc
a continuous portion of a circle
asymptote
a straight line that is the limiting value of a curve
compose
form the substance of
composition
the way in which someone or something is put together
constant
a number representing a quantity with a fixed value
coordinate system
a system that uses coordinates to establish position
cosine
ratio of the adjacent side to the hypotenuse of a right-angled triangle
cube root
a number that when multiplied three times equals a given number
domain
a set of possible values of the independent variable
equation
a mathematical statement that two expressions are the same
exponent
notation of how many times to multiply a quantity by itself
exponential
involving a quantity being multiplied by itself

exponential decay
a decrease that follows an exponential function
expression
a group of symbols that make a mathematical statement
factor
an integer that can be exactly divided into another integer
factorization
breaking down an integer or polynomial into its divisors
Fibonacci sequence
a sequence of numbers in which each number equals the sum of the two preceding numbers
formula
a group of symbols that make a mathematical statement
function
a mathematical relation associating elements between sets
graph
a visual representation of the relations between quantities
growth
a process of becoming larger or longer or more numerous
input
a component of production, such as raw materials or labor
intercept
the point at which a line intersects a coordinate axis
interval
a set containing all points between two given endpoints
inverse function
a function obtained by expressing the dependent variable of one function as the independent variable of another; f and g are inverse functions if f(x)=y and g(y)=x
invertible
having an **additive** or multiplicative inverse

linear
involving an equation whose terms are of the first degree
logarithm
the exponent required to produce a given number
logarithmic
of or relating to or using logarithms
maximum
the point on a curve where the tangent changes
midline
the median plane of the body (or some part of the body)
minimum
the point on a curve where the tangent changes
model
a hypothetical description of a complex entity or process
negative
less than zero
notation
a technical system of symbols to represent special things
output
production of a certain amount
parameter
a constant in the equation of a curve that can be varied
period
the interval to complete one cycle of a repeating phenomenon
periodic
happening or recurring at regular intervals
periodicity
the quality of recurring at regular intervals
polynomial
a mathematical function that is the sum of a number of terms

positive
greater than zero
quadrant
any of the four areas into which a plane is divided
quadratic
of or relating to the second power
radian
the unit of plane angle adopted under the Systeme International d'Unites; equal to the angle at the center of a circle subtended by an arc equal in length to the radius (approximately 57.295 degrees)
range
set of values of a variable for which a function is defined
rate
a quantity considered as a proportion of another quantity
rational
capable of being expressed as a quotient of integers
recursive
characterized by repetition
sequence
serial arrangement in which things follow in logical order
sine
ratio of the length of the side opposite the given angle to the length of the hypotenuse of a right-angled triangle
square root
a number that when multiplied by itself equals a given number
subtend
be opposite to; of angles and sides, in geometry
symmetry
balance among the parts of something
table
a set of data arranged in rows and columns

tangent
ratio of the opposite and adjacent sides of a right triangle
translate
change the position of in space without rotation

trigonometric function
function of an angle expressed as a ratio of the length of the sides of right-angled triangle containing the angle
zero
the mathematical symbol 0 denoting absence of quantity